A model for atmospheric circulation

B S Lakshmi
JNTU College
Of Engineering
Hyderabad

K L Vasundhara
Vidya Jyothi Institute
of Technology
Hyderabad

In this paper we study a mathematical model for atmospheric circulation. If the wind directions of the atmosphere were averaged over large sections of the Earth's surface, a circulation pattern evolves – developed by the heating of the sun and also due to the Earth's rotation. To describe this complex weather system more than 10^5 variables would be involved. We propose a low order model based on a system proposed by Lorenz [4], which models a modified Hadley circulation. (This model will be referred to as the Lorenz-84 Model.) The model involves three dependent variables : x representing the poleward temperature gradient, y and z representing the cosine and sine phases of a sequence of eddies which are large-scale and superposed and t, the independent variable representing time.

Keywords: circulation, Lorenz, attractors

Introduction

The Earth's atmosphere is in constant circulation due to the Earth's atmosphere being heated by the Sun and the Earth's rotation. Mega scale circulatory events develop in the atmosphere because of the uneven heating of the surface of the earth.A succession of heating of the air near the Earth's surface, rising and cool air coming down sets up a general circulation pattern: air rises near the equator, moves north and south away from the equator at higher altitudes, sinks down near the poles, and flows back along the surface from both poles to the equator. This type of flow is called Hadley circulation after George Hadley, who was the first to describe the process [8]. We show a figure depicting Hadley circulation:

To describe this complex weather system more than 10^5 variables would be involved. Computations based on these variables would then be extremely complex and costs of computations,immense. We examine a low order model which models a modified Hadley circulation which is drawn from a paper by Lorenz [4].

$$\begin{aligned}
\frac{dx}{dt} &= -y^2 - z^2 - ax \\
\frac{dy}{dt} &= xy - bxz - y \\
\frac{dz}{dt} &= bxy + xz - z
\end{aligned} \tag{1}$$

where,t,the independent variable represents time,x represents the pole ward temperature gradient or the intensity of the westerly wind current,or in other words it stands for the strength of a symmetric, globally averaged westerly wind current.y and

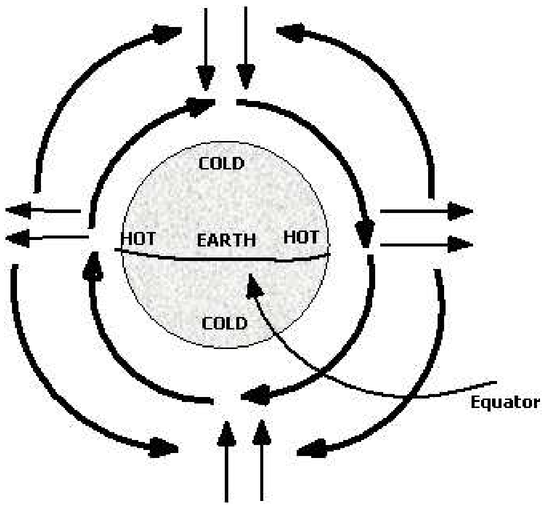


Figure 1: Hadley Circulation.

z represent the cosine and sine phases of a sequence of eddies which are large-scale and super posed. The variables y and z are the strengths of cosine and sine phases of a chain of superposed waves transporting heat poleward. The terms in b represent displacement of the waves due to interaction with the westerly wind. The coefficient a, if less than 1, allows the westerly wind current to damp less rapidly than the waves.

Other authors have worked on this model. For instance Broer et.al in [1] study a particular case of periodic forcing of parameters F and G where the terms in F and G are thermal forcings: F represents the symmetric cross-latitude heating contrast and G accounts for the asymmetric heating contrast between oceans and continents. Lennaert van Veen in [5] analyzes a model which describes midlatitude atmospheric flow on a synoptic scale, a few thousand kilometers in a space and a week or so in time. He studies a six dimensional model related to the Lorenz-84 model. The focus of this study is on the interaction of the jet stream and the baroclinic waves and its representation in a low order model. With the aid of discretisation in the vertical and Galerkin truncation in the horizontal coordinates, they approximate the equations by a finite number of ordinary differential equations. J. G Friere et. al., in [2] examine phase diagrams detailing the intransitivity observed in the climate scenarios supported by a prototype atmospheric general circulation model, namely, the Lorenz-84 low-order model. These eddies transport heat towards the pole. The terms xy and yz represent amplifications of the eddies through interaction with the westerlies. by their interactions with x or the westerlies. This has been studied by some authors. Numerical and analytical explorations can be found in [6] and [10], a bifurcation analysis in [9]. We propose a slightly different model based on the Lorenz model and examine its implications.

1 Analysis Of The System

As we mentioned in the introduction several authors have studied the Lorenz 84 Model and analyzed its attracting sets from various numerical perspectives. Our intention is not to repeat their analysis, but to understand a simple variation of the Lorenz 84 Model by analyzing what happens to the equations if we neglect the effect of the terms aF and G that is the effect of symmetric and asymmetric thermal forcings. F and G are the values to which X and Y would be driven if the westerlies current and eddies were not coupled. In one case under study of the model we propose, we neglect the terms F and G. We would like to remark that this is equivalent to saying that the westerlies and the eddies were not uncoupled. This is in our opinion not unrealistic.

Of course since the equations are highly nonlinear it is interesting to see how the various attracting sets emerge for different values of the parameters. We present a brief graphical analysis of the effect of the change in the system as far as the attracting sets are concerned with the change of a few parameters. The Lorenz-84 Model is given by equations (2). As mentioned earlier, the terms in F and G are thermal forcings: F is the symmetric cross-latitude heating contrast and G accounts for the asymmetric heating contrast between oceans and continents.

$$\begin{aligned}\frac{dx}{dt} &= -y^2 - z^2 - ax + aF \\ \frac{dy}{dt} &= xy - bxz - y + bG \\ \frac{dz}{dt} &= bxy + xz - z\end{aligned} \tag{2}$$

In our model we consider various cases. Primarily our interest is in the first case where we do not take into consideration the uncoupling of the westerlies with the atmospheric eddy currents.

We begin by determining the equilibrium points of the system, Equation (1). The equilibrium point which is real is obtained to be (0,0,0). Linearizing the system around this equilibrium we obtain the system

$$\begin{aligned}\frac{dx}{dt} &= -ax \\ \frac{dy}{dt} &= -y \\ \frac{dz}{dt} &= -z\end{aligned} \tag{3}$$

The system (3) can be easily solved and has the solution $x = c_1 e^{-at}$, $y = c_2 e^{-t}$ $z = c_3 e^{-t}$ c_1, c_2, c_3 being arbitrary constants. The coefficient matrix of the linearized system at the equilibrium point (0,0,0) is

$$\begin{pmatrix} -a & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

The Eigen values are seen to be $(-a, -1, -1)$. This indicates that the equilibrium is a stable equilibrium. That the equilibrium is stable as is also evident from the solution of the linearized system.

2 Liapunov Function

We now apply Liapunov's direct method to the problem on hand. Let $L(x, y, z) = x^2 + y^2 + z^2$ be a Liapunov function, $L(0, 0, 0) = 0$, $L(x, y, z) > 0$ for $x > 0$, $y > 0$, $z > 0$

$$\begin{aligned}\dot{L} &= x(-y^2 - z^2 - ax) + y(xy - bxz - y) + z(bxy + xz - z) \\ &= -ax^2 - y^2 - z^2 < 0\end{aligned}$$

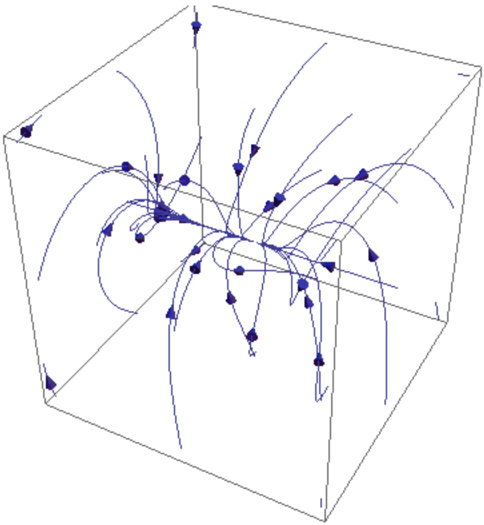
To illustrate this better we include a figure (2) where one can clearly see the direction field going toward the equilibrium point.

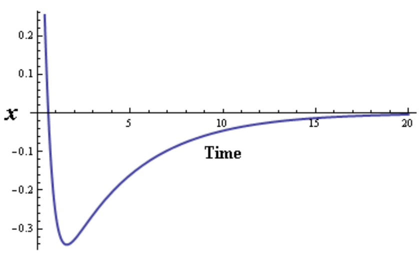
A numerical solution of the system for x ,y and z are obtained and compared with the results obtained by plotting Lorenz's 84-model The figures follow:

It is evident from Fig. 3 that the temperature gradient x, is steep initially and decreases when time t is nearer zero that is the beginning of the cycle, which is the winter season. Then as t increases or the months progress forward, the gradient increases (positive) for some time and finally tapers off.

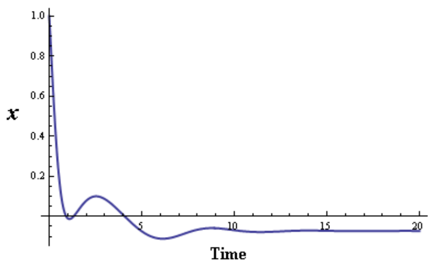
We next show the comparative graphs for the sine (z) and the cosine (y) phases of superposed eddies which transport heat poleward with time t for our model and the Lorenz-84 Model. If we compare the way y (or the cosine phases of superposed eddies which transport heat poleward) changes with the months, in our model $y \to 0$ with time, whereas in Lorenz's model $y \to G$. In

Figure 2: In this picture we show some lines showing the direction of the force field





(a) x- The poleward temperature gradient and time t.



(b) x- The poleward temperature gradient and time t form the Lorenz-84 Model

Figure 3: Comparison of the two models

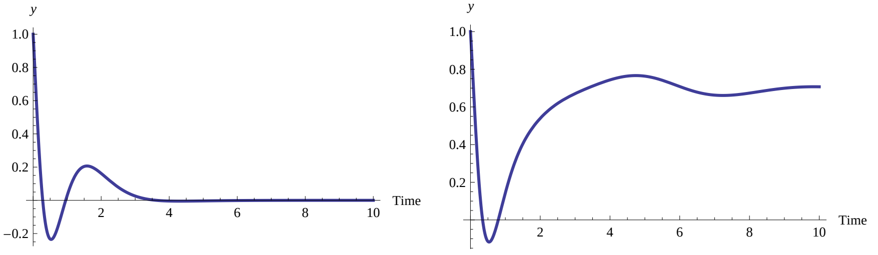


Figure 4: y- The Cosine phase of superposed eddies and t (time)

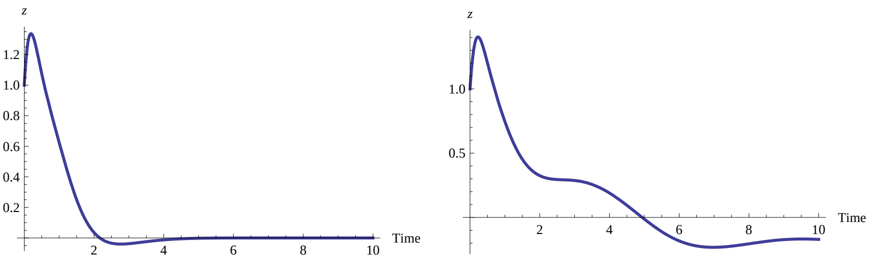


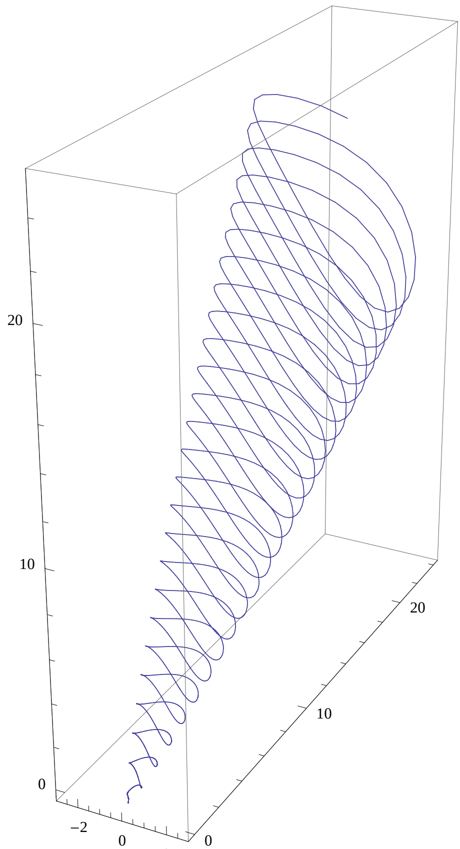
Figure 5: z- The Sine phase of superposed eddies and t (time)

this case G has been chosen to be 0.8.(As chosen by Lorenz in his Lorenz-84 Model)
When we examine the variation of z, in the original Lorenz model the temperature decrease is oscillatory, our analysis on the other hand throws up a smooth decrease. We next plot the solution of our model for x $y$and z. The relation between the terms x, y and z is apparent that is they tend to the equilibrium point with increase in time.The result of our analysis is rather interesting. When the terms F and G are neglected, we see that the poleward temperature gradient namely x is smooth with the passage of the months (t), and not oscillatory as seen in the Lorenz 84 Model. Our result maybe closer to real observations than an oscillating change. This point can be seen in the following real-time temperature profile: [7]. As mentioned earlier, we present a numerical simulation of the Lorenz 84 Model for different values of the parameter b. The figures are simulated for the values of $F = 14.14$, $G = 3.19$, $t = 7.62$, $x_0 = 6.47$, $y_0 = 89.57$, $z_0 = 21.77$. with changing values of b. It can be observed that with the change in the parameter b from $0.4 < b \leq .1$ to $0.1 < b \leq 3.56$, the attracting sets increase from 2 to 3.

It can be observed that with the change in the parameter b from $0.4 < b \leq .1$ to $0.1 < b \leq 3.56$, the attracting sets again change in number. We would like to point out that while relatively wide portions of the $F \times G$ parameter plane have been considered, before by a number of authors, virtually all computations so far were done only for Lorenz's choice of a=0.25 and b=4. The exception are the considerations of van Veen, [5] who studies the case a=0.35 and b=1.33. We have done an analysis for a wide range of values of b from $.45 < b \leq 6.01$.

For the next case we consider a periodic perturbation of the

Figure 6: In this picture we show the relation between x, y and z



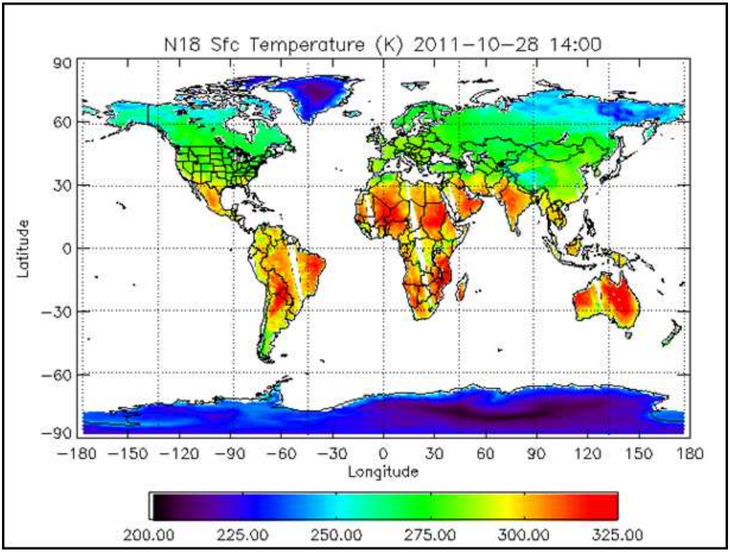
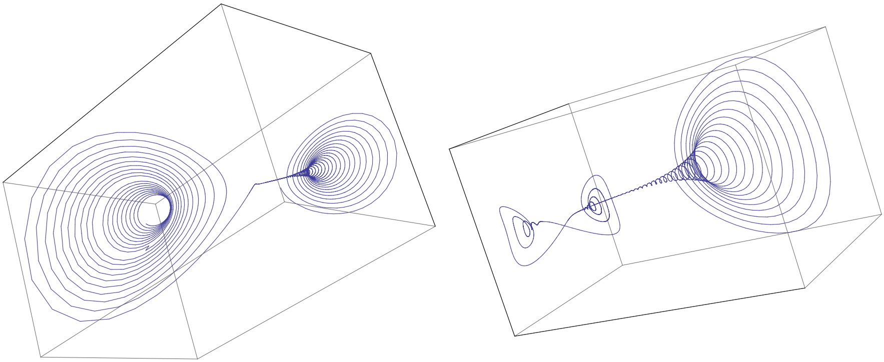


Figure 7: Latitudinal and Longitudinal variation of Temperatures (NOAA-2011)



(a) The Lorenz84 Model with $0.4 < b \leq .1$

(b) The Lorenz84 Model with $0.1 < b \leq 3.56$

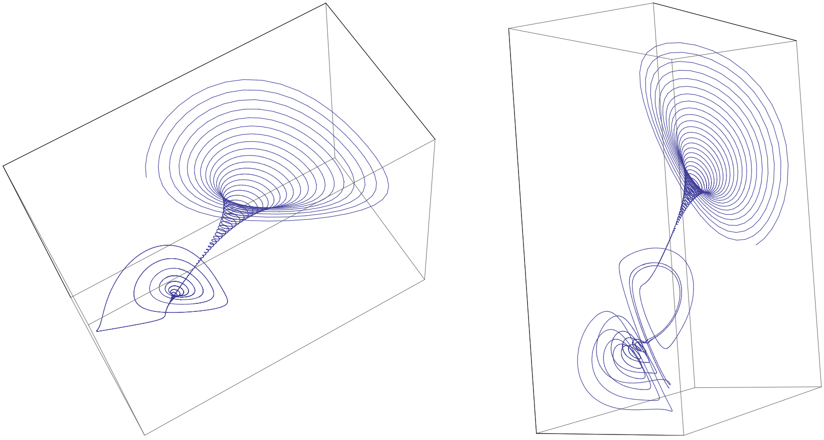
Figure 8: Lorenz 84 Model for different values of b

parameters F and G with a multiplicative term, $\cos(rt)$ so the system under consideration is

$$\begin{aligned} \frac{dx}{dt} &= -y^2 - z^2 - ax + aF\cos(rt) \\ \frac{dy}{dt} &= xy - bxz - y + bG\cos(rt) \\ \frac{dz}{dt} &= bxy + xz - z \end{aligned} \quad (4)$$

A numerical simulation of equation (4) is

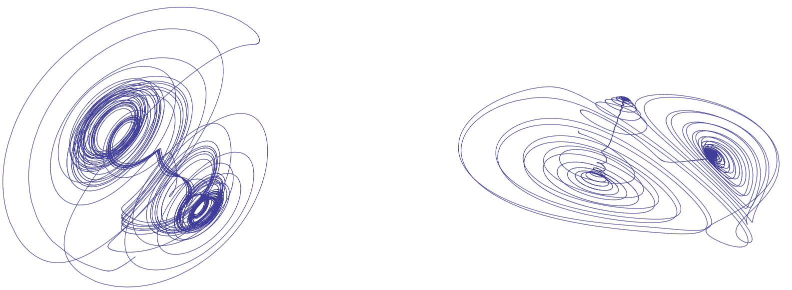
In Figure (10) the values of the parameters are as follows: b=18.57, G=2.97, a= 20.33, r=5.,1, t=17.95. Two attractors are clearly visible for two different values of F.



(a) The Lorenz84 Model with $3.56 < b \leq 5.1$ (b) The Lorenz84 Model with $5.1 < b \leq 5.8$

Figure 9: Lorenz 84 Model for different values of b

We next Simulate a model of Lorenz 84 with a different kind of Periodic Perturbation, that is $(1 + e\cos(rt))$, which we show for the values for t=8.27, b=4,f=7, G=1.7545, e= .01, a =.25, r= .086.



(a) The Lorenz84 Model with a periodic perturbation with $F = 4.95$

(b) The Lorenz84 Model with a periodic perturbation with $F = 14.4$

Figure 10: The perturbed Lorenz 84 Model for different values of F

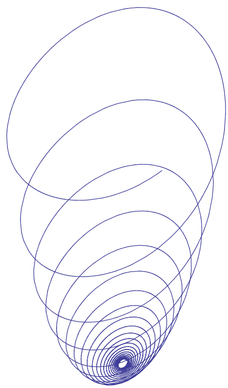


Figure 11: A Phase-portrait of a periodically perturbed Lorenz-84 system , Pertubative term= $(1 + e\cos(rt))$

3 Conclusion

We started with the Lorenz 84 model, we then modified the model by neglecting the effect of the decoupling of the westerlies and the atmospheric eddies. In this new formulation we find that the poleward temperature gradient first increases with time and then steadily and smoothly decreases till it attains asymptotically the equilibrium value. This conclusion is borne out by a dynamical system analysis, including the Liapunov function. Whereas in the original Lorenz model the temperature decrease is oscillatory contrary to observations, our analysis on the other hand throws up a smooth decrease consistent with observations. We also carried out some numerical simulations for the Lorenz 84 Model for a range of values of the parameters including two different periodic perturbative coefficients. The results show extremely chaotic behaviour, with the presence of multiple attractors which change in number for a small change in the parameters.

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