A Further Analysis of the Blackbody Radiation

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A review of the classical deduction of the Rayleigh-Jeans formula is done. By using the same procedure but substituting the Abraham-Lorentz equation by the Eliezer-Ford equation, we derive another expression for the frequency energy density of a blackbody. In the new formulation the new frequency energy density does not diverge, even though the ultraviolet catastrophe still appears because the integral over the frequency is infinite. Thus, the consequences of the new formulation are discussed and its uselfulness is analyzed.

Keywords: Blackbody, steady state, reaction force.

Introduction

Classical thermodynamics was developed in the middle of the

19th century. One of the most important creator or contributor of the theory was Kirchhoff. During this time, the engineering were very interested in obtaining new metallurgy techniques and one of the problems consisted of knowing the exact temperature of an oven without interfering in the industrial process. Indeed, the knowledge of the temperature of an industrial oven represented one of the biggest researches in the Germany of the 19th century. But how measure it without interfering in the process? The idea was to make a small hole in order to have a spectrum of the radiation coming from the oven and to try of linking it with its temperature; that is: the problem of the blackbody radiation appeared. By using the incipient thermodynamics, in particular the second law, Kirchhoff was the first scientist in studying the blackbody radiation. A key contribution derived by Kirchhoff [1] was that the radiation of a blackbody is independent of the medium surrounding it (nature of the walls). Many famous scientists were interested in the problem. Indeed, in 1879, Stefan [1], and in 1884, Boltzmann [1] by using classical electrodynamics, they proposed that the total radiated energy produced by a blackbody was proportional the four power of the absolute temperature; that is:

$$U(T) = aVT^4. \tag{1}$$

Then, in 1896, Wien [2] proposed an exponential law which can be described by

$$u_v(T) = \alpha v^3 \exp\left[-\lambda \frac{v}{T}\right]. \tag{2}$$

Nevertheless, in Berlin, in the Physikalisch Technische Reichsanstalt, in 1900, Lummer and Pringsheim [3, 4] concluded that

Wien law was incorrect for higher frequencies. Rubens and Kurlbaum [3, 4], by analyzing the infrared spectrum, they arrived to a similar conclusion. However, Kurlbaum [3, 4] communicated to Planck, whom had recently replaced Kirchhoff, that the Wien law for small frequencies failed and the frequency energy density was proportional to the temperature. Finally, Planck [2, 3] corrected the Wien law by proposing the following expression:

$$u_v(T) = \frac{8\pi h v^3}{c^3} \frac{1}{\exp\left[\frac{hv}{kT}\right] - 1}. \tag{3}$$

A few time later, Planck [4] was able to explain his formula by proposing a quantization of the energy. First, he used the Kirchhoff result that the spectrum of a blackbody was independent of the nature of the walls and he proposed that a system composed by electric dipoles may represent a blackbody. Secondly, by using retarded and advanced electromagnetic fields, he deduced a reaction force which coincides with what Abraham [5] had obtained at the same time. He also obtained the same reaction force by considering an average of the motion of an oscillating charged particle. Planck [4] used this result in order to derive the steady state of electric dipoles (harmonic oscillators) immersed in an electric field. As a consequence of it, the Rayleigh-Jeans law and the ultraviolet catastrophe was predicted by classical mechanics. Therefore, he proposed to quantized the energy of the oscillator and finally he justified his formula which represents the beginning of quantum mechanics. However, even if Planck obtained the Rayleigh-Jeans result by using his reaction force, his deduction presents a confusion about the interpretation of the frequency and his result is just a mere coincidence.

The paper is organized as follows: in the second section, we expose a summary of the Planck deduction. We used the Eliezer-Ford [6, 7] equation in order to obtain a correction to the Raleigh-Jeans formula in third section. Even though the new expression of the frequency energy density does not diverge for high frequencies, the ultraviolet catastrophe still exists. In fourth section , we analyze Planck procedure for obtaining the ultraviolet catastrophe and we conclude that his result does not exactly describe the Rayleigh-Jeans law. In the conclusion, fifth section, some concluding remarks are done.

The classical ultraviolet catastrophe

Let us make a simple review of the classical ultraviolet catastrophe. It is a well known fact that Kirchhoff demonstrated that the blackbody spectral density radiation is independent of the nature of the wall cavity. Therefore, as Planck proposed, we can substitute the walls of the cavity by a set of small independent oscillators in one dimension. Taking into account the Abraham-Lorentz [5] equation, we have:

$$\ddot{x} + w^2 x - \frac{2e^2}{3mc^3}\dddot{x} = \frac{e}{m}\varepsilon_x, \tag{4}$$

where $w = \sqrt{k/m}$, and e, m, k and ε_x represent the charge, the mass, the restitution constant and the x-component of the electric field in the cavity, respectively. Considering just the steady-state and by using Fourier transformations, we obtain

$$\widetilde{x}(\Omega) = \frac{e}{m}\frac{\widetilde{\varepsilon_x}(\Omega)}{w^2 - \Omega^2 + i2e^2\Omega^3/3mc^3}. \tag{5}$$

On the other side, the mean energy of an oscillator is

$$\begin{aligned} \overline{E}^{\,t} &= \overline{T+V}^{\,t} = 2\overline{T}^{\,t} = m\,\overline{\dot{x}^{2}}^{\,t} \\ &= \frac{2e^2}{m}\int_0^\infty \frac{\Omega^2\left|\tilde{\varepsilon}_x(\Omega)\right|^2}{\left[w^2-\Omega^2\right]^2+\left[2e^2\Omega^3/3mc^3\right]^2}. \end{aligned} \tag{6}$$

Since there is a maximum in $w = \Omega$ (as a delta function, Lorenz function), we can integrate it just by substituting the value of Ω by w whenever $w - \Omega$ does not appears. Therefore, $(\xi = w - \Omega)$

$$\begin{aligned} \overline{E}^{\,t} &= \frac{2e^2}{m}\left|\varepsilon_x(w)\right|^2 \int \frac{d\xi}{\xi^2+(2e^2w^2/3mc^3)^2} \\ &= \frac{3\pi c^3}{2w^2}\left|\varepsilon_x(w)\right|^2. \end{aligned} \tag{7}$$

Since the mean energy in the cavity is given by

$$\begin{aligned} u &= \frac{1}{4\pi}\overline{\varepsilon^2}^{\,t} = \frac{1}{4\pi}\overline{\varepsilon_x^2+\varepsilon_y^2+\varepsilon_z^2}^{\,t} = \frac{3}{4\pi}\overline{\varepsilon_x^2}^{\,t} \\ &= \frac{3}{2\pi}\int_0^\infty \left|\bar{\varepsilon}_x(w)\right|^2 dw = \int_0^\infty \rho(w)dw. \end{aligned} \tag{8}$$

By substituting this last result in Eq. (7), we arrive at

$$\rho(w,T) = \frac{w^2}{\pi^2c^3}\overline{E(w)}^{\,t}. \tag{9}$$

By using classical statistical theory, let us now calculate the mean energy $\overline{E}$. The number of particles with energy E at temperature T is equal to

$$N(E) = N_o \exp\left[-\beta E\right], \tag{10}$$

where $\beta = \frac{1}{kT}$ being k the regular Boltzmann constant. Therefore the mean energy can be calculated as follows:

$$\begin{aligned}\overline{E} &= \frac{\int_0^\infty E \exp\left[-\beta E\right] dE}{\int_0^\infty \exp\left[-\beta E\right] dE} = -\frac{\partial}{\partial \beta} \ln \int_0^\infty \exp\left[-\beta E\right] dE \\ &= \frac{\partial}{\partial \beta} \ln \beta = \frac{1}{\beta} = kT \end{aligned} \tag{11}$$

On the other hand by using the ergodic theorem,

$$\overline{E}^{\,t} = \overline{E}, \tag{12}$$

we obtain:

$$\rho(w, T) = \frac{w^2}{\pi^2 c^3} kT. \tag{13}$$

This is known as the Rayleigh-Jeans formula. If we calculate the total energy, that is,

$$u = \int_0^\infty \rho(w, T) dw, \tag{14}$$

we obtain the famous ultraviolet catastrophe since the energy diverges. Moreover, we can compare the Rayleigh-Jeans result with the experimental data or the Planck proposal [3] and we can notice that the for small frequencies both results coincide but for higher frequencies the Rayleigh-Jeans formula diverges (see Fig. 1).

Eliezer-Ford Equation

Let us now try to obtain the spectral density of a blackbody by using the Eliezer-Ford [6, 7] equation. Eq. (4) which it can

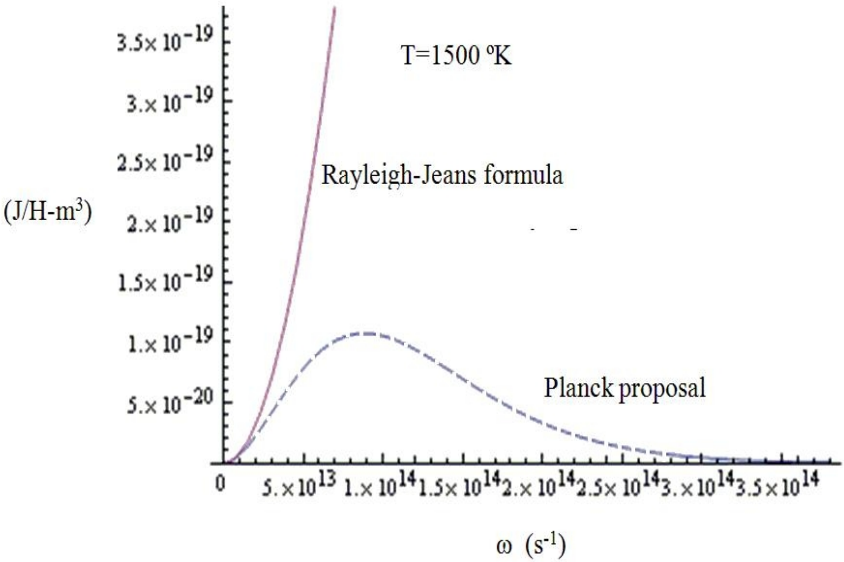

Figure 1: Rayleigh-Jeans Formula vs Planck Proposal

be expressed in this case by

$$\ddot{x} + w^2 x - \frac{2e^2}{3mc^3} w^2 \dot{x} = \frac{e}{m}\varepsilon_x + \frac{2e^2}{3mc^3}\frac{e}{m}\dot{\varepsilon}_x. \tag{15}$$

And Eq. (5) is now

$$\begin{aligned} \widetilde{x}(\Omega) &= \frac{e}{m}\frac{\widetilde{\varepsilon}_x(\Omega)\left[1 + i\frac{2e^2}{3mc^3}\Omega\right]}{w^2 - \Omega^2 + i2e^2\Omega^3/3mc^3} \\ &= \frac{e}{m}\frac{\widetilde{\varepsilon}_x(\Omega)\left[1 + i\tau_o\Omega\right]}{w^2 - \Omega^2 + i\tau_o w^2\Omega}, \end{aligned} \tag{16}$$

where we have introduced $\tau_o = \frac{2e^2}{3mc^3}$. Eq. (6) turns on

$$\begin{aligned} \overline{E\ ^t} &= \overline{T + V\ ^t} = 2\overline{T\ ^t} = m\,\overline{\dot{x}^2\ ^t} \\ &= \frac{2e^2}{m}\int_0^\infty \frac{\Omega^2\left|\widetilde{\varepsilon}_x(\Omega)\right|^2\left[1 + \tau_o\Omega\right]^2}{\left[w^2 - \Omega^2\right]^2 + \left[2e^2 w^2\Omega/3mc^3\right]^2}. \end{aligned} \tag{17}$$

Finally, we obtain

$$\begin{aligned} \overline{E\ ^t} &= \frac{2e^2}{m}\left|\widetilde{\varepsilon}_x(w)\right|^2\left[1 + \tau_o^2 w^2\right]\int\frac{d\xi}{\xi^2 + \tau_o^2 w^4} \\ &= \frac{3\pi c^3}{2w^2}\left|\widetilde{\varepsilon}_x(w)\right|^2.\left[1 + \tau_o^2 w^2\right]. \end{aligned} \tag{18}$$

Therefore, we arrive at

$$\rho(w, T) = \frac{2w^2}{\pi^2 c^3}\frac{\overline{E(w)\ ^t}}{1 + \tau_o^2 w^2}. \tag{19}$$

Finally, (see Fig. 2)

$$u = \frac{2w^2}{\pi^2 c^3} \frac{kT}{1 + \tau_o^2 w^2}. \tag{20}$$

The ultraviolet catastrophe does not disappear since the integral $\int udw$ still diverges.

Nevertheless, it has to be pointed out that the maximum of the Eliezer-Ford formula coincides with the Planck's one just when the temperature is closed to 10^{13} K (see Fig. 2).

Planck Confusion

The Rayleigh-Jeans formula can be derived by considering an electromagnetic field in a metallic cavity. The electromagnetic modes are considered and if the cavity is enough big, we can always consider that the frequencies are continuous in order to obtain a density of states for the frequencies

$$n(w) = \left[\frac{w}{c}\right]^2 \frac{V}{\pi^2 c}. \tag{21}$$

The Rayleigh-Jeans formula is obtained just by considering that the energy density is

$$\rho(w,T) = \frac{U}{V} = \frac{n(w)}{V}\overline{E(w)}^{\;t} = \left[\frac{w}{c}\right]^2 \frac{kT}{\pi^2 c} = \frac{w^2}{\pi^2 c^3} kT, \tag{22}$$

It has to be pointed out that a small cavity will not represent a blackbody radiation since many frequencies will not be allowed and the density number of frequency will not coincides with Eq. (21). For practical uses, the sun and an oven can be considered

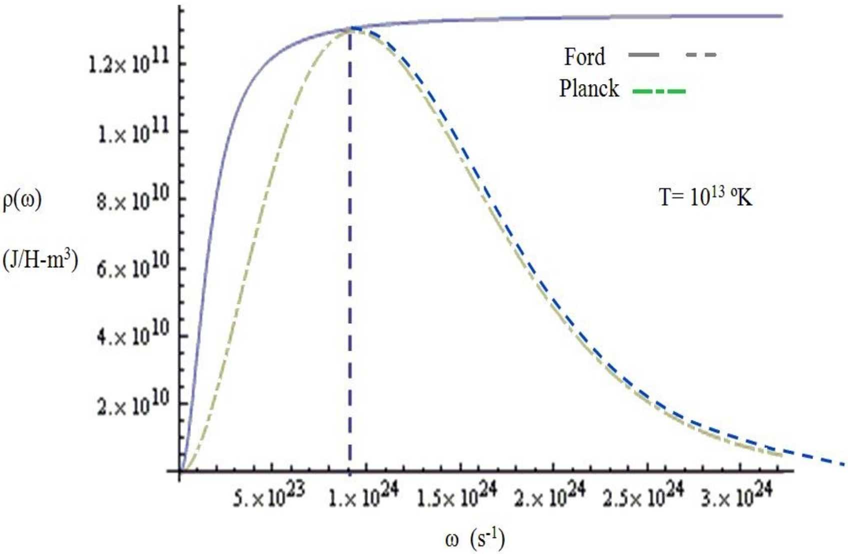

Figure 2: Density of radiation by using the Eliezer-Ford equation

as a blackbody. So, within the classical theory ($\frac{hv}{kT} \ll 1$) the Rayleigh-Jeans is correct.

Nevertheless, a question immediately emerges why if Eliezer-Ford equation is a better representation of the motion of a charged particle, the result differs from the Rayleigh-Jeans theory? However, the difference between the two results generated by the two equations is very small for $\frac{hv}{kT} \ll 1$ and it cannot detected. Nevertheless, since it represents an absurd though that a set of dipoles generated an infinite energy, we can think that something has to be corrected in the theory. In first instance, we can think that, by considering relativistic effects, Rayleigh-Jeans formula will be changed in order to obtain a closer result to the Planck formula; that is: for higher frequencies, the accelerations of the charges will be bigger and consequently relativity will impact on the result and the energy density may decrease for higher frequencies as happened for the Eliezer-Ford equation case. When Lorentz-Dirac [8] and Landau-Lifshitz [9] equation are considered in cero order in the velocity, we obtain the Abraham-Lorentz and Eliezer-Ford equations, respectively. Therefore, a first correction will include the first order approximation terms in the velocity. Let us consider the first order corrections in the equation of motions and add them in Eqs. (4) and (15), that is: for the Abraham-Lorentz case, we must include

$$\tau_o m \left[\frac{3\dot{\overrightarrow{x}} \cdot \ddot{\overrightarrow{x}}}{c^2} \right] \ddot{\overrightarrow{x}}, \tag{23}$$

and for the Eliezer-Ford case,

$$\tau_o \left[\frac{3 \dot{\overrightarrow{x}} \cdot \overrightarrow{F}}{c^2} \right] \frac{\overrightarrow{F}}{m}. \tag{24}$$

In both cases, considering that the charged particle is submitted to a harmonic force and to an oscillating electric field, the obtained equations will not be linear. Consequently, due to the non resonance effect the steady state will not be reached. Nevertheless, if we average the correction terms, Eqs. (23) and (24), we notice that the contribution will vanish as a first approximation. Therefore, the results described in Eqs. (13) and (20) will not suffer a variation. Finally, it seems that Planck method for obtaining the Rayleigh-Jeans formula is correct in classical mechanics. But if we look at the frequency that it is considered in Eqs. (13) and (20), we will notice that it represents the frequency of the dipole, that is: $w = \sqrt{\frac{k}{m}}$. Therefore there is no sense to consider the density energy as dependent of all the frequencies since there is just one. Thus, the ultraviolet catastrophe is not obtained because we cannot integrate over all the frequency since just one of them exists. This means that the set of electric dipole does not describe a blackbody. This was evident because there will be just one resonance frequency and the spectrum will be composed of just one frequency. Therefore, any radiation that will arrive to such a body will not be absorbed, and as a consequence we will not have a blackbody. The statistical result of identifying $\overline{E^{\ t}} = \overline{E} = kT$, does not have any significance. The result is just a mere coincidence. We will be able to change the Planck method by considering a set of dipoles with all the possible frequencies, which is physically impossible. Indeed, if we want to deal with oscillators, we will be obligated to consider a gas of phonons which will have all the possible frequencies. Also, the energies will be quantized, but this topic belongs to quantum mechanics.

Conclusion

We have demonstrated that the way of obtaining the Rayleigh-Jeans formula cannot be reached by using a set of dipoles with the same fundamental frequency and that even if Planck result is formally correct, its represents the steady state of a resonant frequency $w = \sqrt{\frac{k}{m}}$ with the electric field. That is, Eq. (9) must be written as

$$\rho(\sqrt{\frac{k}{m}}, T) = \frac{\frac{k}{m}}{\pi^2 c^3} E(\sqrt{\frac{k}{m}}). \tag{25}$$

Eq. (25) shows that a set of dipoles with the same frequencies does not represents a blackbody. Therefore, the deduction of the Rayleigh-Jeans formula based on a set of dipoles with the same frequencies cannot be reached. Nevertheless, Rayleigh-Jeans formula can always be deduced by dealing with a metallic cavity which contains a electromagnetic field and considering the constraint in the walls.

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References

[1] D. F. Lawden, *Principles of Thermodynamics and Statistical Mechanics*, Dover Publications Inc, New York, 2005.

[2] M. Planck, *On an Improvement of Wien Equation for the Spectrum*, Verh. Dtsch. Phys. Ges., Berlin, 2, 202, 1900.

[3] M. Planck, *On the theory of Energy Distribution Law of the Normal Spectrum*, Verh. Dtsch. Phys. Ges., Berlin, 2, 237, 1900.

[4] M. Planck, *The Theory of Heat Radiation*, Dover Publications Inc., New York, 2nd. Ed.,1956.

[5] F. Rohrlich, *Classical Charged Particles*, Addison-Wesley, Redwood City, 1965.

[6] C. Jayaratnam Eliezer, Proc. Roy. Soc. London A 194, 543-555, 1948.

[7] G. W. Ford, R. F. O'Connell, Phys. Lett. A 174, 182-184, 1993.

[8] P. A. M. Dirac, Proc. Roy. Soc. London A 167, 148169, 1938.

[9] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* Pergamon, London, 2nd. Ed., 1962. §76.