A New Approach to General Relativity

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Here we present a new point of view for general relativity and/or space-time metrics that is remarkably different from the well-known viewpoint of general relativity. From this unique standpoint, we attempt to derive a new metric as an alternative to the Schwarzschild metric for any planet in the solar system. After determining the metric by means of some simple mathematical and physical manipulations, we used this alternative metric to recalculate the perihelion precession of any planet in the solar system and deflection of light that passes near the sun, as examples of this new viewpoint. While we obtained the result of classical general relativity for the perihelion procession, we found a slightly different result, relative to classical general relativity, for the deflection of light.

Keywords: General relativity, Schwarzschild metric, basis vectors, perihelion precession, deflection of light

1) Introduction: Origins of a New Metric

As is shown in [1], the main equations of general relativity, known as the Einstein field equations, can be expressed using only four basis vectors, or tetrads. Additionally, it is seen that the Einstein tensor in terms of tetrads has the same form as the stress-energy tensor of electromagnetism/dynamics and the tetrads satisfy

$$\partial^{\alpha}\partial_{\alpha}\mathbf{e}_{\nu} = \mathbf{j}_{\nu} . \qquad (1)$$

As stated in [1], (1) is not only a differential equation but also a mathematical rule which basis vectors of any metric must satisfy. Since every metric can be expressed in terms of some basis vectors, (1) also brings some certain limits for all metrics. As a result of this fact it is expected that all basis vectors of all metrics must satisfy (1). Subsequently, if there is such a rule that basis vectors must obey, it is expected to check this condition for the known cases. Obviously, the best-known case is the Schwarzschild solution, which will be checked in our study.

The Schwarzschild solution is the most famous exact solution of the Einstein equations, thus, it has an exclusive place and a crucial role in general relativity. It can correctly predict the perihelion procession of a planet in the solar system, and can give a close prediction to observations for the deflection of light that passes near the sun. In addition, some mysterious concepts such as black holes or horizon problems can be investigated by means of the Schwarzschild metric.

Despite this, however, there is a significant discrepancy between the Schwarzschild metric and [1]. The Schwarzschild metric was originally derived for a planet orbit around the sun, so it is expected that basis vectors of the Schwarzschild metric must satisfy the vacuum case of (1), that is $\partial^{\alpha}\partial_{\alpha}\mathbf{e}_{\nu} = \mathbf{0}$ (because there is no mass density and mass flux outside of the sun for the solar system) or equivalently $\partial^{\alpha}\mathbf{j}_{\alpha} = \mathbf{0}$. Since the Schwarzschild metric is

$$d\sigma^2 = -c^2\left(1-\frac{2GM}{c^2r}\right)dt^2 + \left(1-\frac{2GM}{c^2r}\right)^{-1} dr^2$$
$$+r^2d\theta^2 + r^2Sin^2\theta d\varphi^2,$$

the basis vectors will be

$$\mathbf{e}_0^* = \left(1-2GM/c^2r\right)^{1/2}\mathbf{n}_0,$$

$$\mathbf{e}_r^* = \left(1-2GM/c^2r\right)^{-1/2}\mathbf{n}_r, \qquad \text{(x33q)}$$

where $\mathbf{n}_0$ and $\mathbf{n}_r$ are unit vectors.

From the last two equations it can easily be seen that $\partial^\alpha\partial_\alpha\mathbf{e}_0^* \neq \mathbf{0}$ and $\partial^\alpha\partial_\alpha\mathbf{e}_r^* \neq \mathbf{0}$. This suggests that the Schwarzschild metric is not a proper metric according to the point of view of [1] and a new metric for the solar system whose basis vectors satisfy (1) needs to be found. Hence, this was the first basis for searching a new metric as an alternative to the Schwarzschild metric.

Besides the above mathematical necessity, we can also give a physical condition for finding a new metric. The foundation of this condition proposes a new and significant limit condition between the space-time metric and classical mechanics. For this limit condition, the metric in terms of tetrads for the most general case should be written first. As seen from (2) in [1], all basis vectors are perpendicular to each other, the metric in terms of tetrads for the most general case is $\left(x^0 \equiv ict\right)$

$$ds^2 = -c^2\mathbf{e}_0\bullet\mathbf{e}_0dt^2 + \mathbf{e}_1\bullet\mathbf{e}_1dx^2 + \mathbf{e}_2\bullet\mathbf{e}_2dy^2 + \mathbf{e}_3\bullet\mathbf{e}_3dz^2. \qquad (2)$$

If (2) is divided by $-c^2d\tau$, where τ is the proper time, it can be written as

$$-\frac{1}{c^2}\left(\frac{ds}{d\tau}\right)^2 = \mathbf{e}_0 \bullet \mathbf{e}_0 \left(\frac{dt}{d\tau}\right)^2 - \mathbf{e}_1 \bullet \mathbf{e}_1 \frac{v_1^2}{c^2} - \mathbf{e}_2 \bullet \mathbf{e}_2 \frac{v_2^2}{c^2} - \mathbf{e}_3 \bullet \mathbf{e}_3 \frac{v_3^2}{c^2}, \quad (3)$$

where $v_1 = \frac{dx}{d\tau}$, $v_2 = \frac{dy}{d\tau}$ and $v_3 = \frac{dz}{d\tau}$. Allowing

$$\frac{dS^2}{d\tau^2} = -\frac{1}{c^2}\left(\frac{ds}{d\tau}\right)^2$$

(3) to become

$$\frac{dS^2}{d\tau^2} = \mathbf{e}_0 \bullet \mathbf{e}_0 \left(\frac{dt}{d\tau}\right)^2 - \mathbf{e}_1 \bullet \mathbf{e}_1 \frac{v_1^2}{c^2} - \mathbf{e}_2 \bullet \mathbf{e}_2 \frac{v_2^2}{c^2} - \mathbf{e}_3 \bullet \mathbf{e}_3 \frac{v_3^2}{c^2}. \quad (4)$$

Here, (4) is a relativistic expression and for classical cases it is known that $\frac{v_i}{c} \ll 1$ ($i = 1, 2, 3$), consequently $\left(\frac{v_i}{c}\right)^2 \approx 0$ can be taken. Although there are inner products of basis vectors, these do not disturb this limit condition, especially for planets in the solar system. Thus (4) can be written as

$$\frac{dS^2}{d\tau^2} \approx \mathbf{e}_0 \bullet \mathbf{e}_0 \left(\frac{dt}{d\tau}\right)^2$$

or

$$dS^2 \approx \mathbf{e}_0 \bullet \mathbf{e}_0 dt^2. \quad (5)$$

On the other hand, it is known that from classical mechanics

$$dS = L(v_i, q_i; t)dt \quad (6)$$

where S is the action function, L is the Lagrangian, v_i are velocities and q_i are coordinates ($i = 1, 2, 3$).

Now we will make the limit assumption: If (5) and (6) are related with classical cases, we assume that actions functions in (5) and (6) are the same action functions, or at least, are proportional with each other. Then we can write

$$L^2(v_i, q_i; t) = L_0^2 \mathbf{e}_0 \bullet \mathbf{e}_0$$

where L_0 is a constant proportion coefficient and it must be equal to $m_0 c^2$ in order to obtain flat space-time metric for a stationary object whose rest mass is m_0, and L is the relativistic Lagrangian for the most general case. But for small speeds L becomes equal to the classical mechanics' Lagrangian. So in the light of this limit condition, (2) can be written as

$$ds^2 = -c^2 \left(\frac{L}{m_0 c^2} \right)^2 dt^2 + \mathbf{e}_1 \bullet \mathbf{e}_1 dx^2 + \mathbf{e}_2 \bullet \mathbf{e}_2 dy^2 + \mathbf{e}_3 \bullet \mathbf{e}_3 dz^2 , \qquad (7)$$

where now m_0 denotes the rest mass of the any planet around sun.

It can be easily seen from (7) that the time component of the Schwarzschild metric is not equal to $\left(\frac{L}{m_0 c^2} \right)^2$ for a planet in the solar system, but approximately equal to $1 + \frac{2U}{m_0 c^2} \cong \left(1 + \frac{U}{m_0 c^2} \right)^2$, where U is the potential energy of the planet. Thus, there is also a need of the finding a new metric whose time component of it is equivalent to $\left(\frac{L}{m_0 c^2} \right)^2$.

So far we have presented our two motivations for finding a new metric. One motivation is based on a mathematical calculation and the other is based on a physical reason. In the remainder of the study we will attempt to find this new metric for any planet in the solar system, and calculate the mathematical results of it. After we find the new metric for the solar system, we will present two well-known examples for it: perihelion precession of a planet and deflection of light. As is known, these two examples are well-known examples used to test the Schwarzschild metric or classical general relativity. Therefore, these examples are chosen to compare our new metrics results with results from classical general relativity. It is expected that the metric, which is derived as a result of the above-mentioned motivations, will give more satisfying results, or at least the same results as those from the Schwarzschild metric. Interpretation of these results will be our last task in this study, although some interpretations will be left for future studies.

2) Determination of the New Metric and Applications of It

Due to spherical symmetry in the solar system due to we can write the metric as

$$ds^2 = -c^2 \mathbf{e}_0 \bullet \mathbf{e}_0 dt^2 + \mathbf{e}_r \bullet \mathbf{e}_r dr^2 + \mathbf{e}_\theta \bullet \mathbf{e}_\theta r^2 d\theta^2 + \mathbf{e}_\varphi \bullet \mathbf{e}_\varphi r^2 Sin^2\theta d\varphi^2 .$$

Since the sun is fixed relative to planets and there is no mass change or current, from $\partial^\alpha \partial_\alpha \mathbf{e}_\nu = \mathbf{0}$, we understand that the spatial part of the metric must be flat. This enables us to write the metric as

$$ds^2 = -c^2 \mathbf{e}_0 \bullet \mathbf{e}_0 dt^2 + dr^2 + r^2 d\theta^2 + r^2 Sin^2\theta d\varphi^2$$

and determine the results.

Since $\mathbf{e}_0 \bullet \mathbf{e}_0$ is proportional with Lagrangian, we need to find Lagrangian for a planet around the sun. In order to find the relativistic Langrangian of the planet around the sun, we assume that the planet has no any kinetic energy at $r \to \infty$ (the potential energy is also zero, but our planet possesses $m_0 c^2$ as rest energy), and that it has acquired some kinetic energy when it is placed to its own orbit at $r = r$ (for this case the potential energy is $-\frac{GMm_0}{r}$, where G is the universal gravitational constant and M is the mass of the sun). Then, the kinetic energy of the planet must be equal to the change in the potential energy (we give another way for finding relativistic Lagrangian of a planet around sun in the Appendix I section). Thus,

$$L = m_0 c^2 + \frac{2GMm_0}{r},$$

$$\mathbf{e}_0 \bullet \mathbf{e}_0 = \left(1 + \frac{r_S}{r}\right)^2,$$

where $r_S = 2GM / c^2$. Then the metric can be written as

$$ds^2 = -c^2 \left(1 + \frac{r_S}{r}\right)^2 dt^2 + dr^2 + r^2 d\theta^2 + r^2 Sin^2\theta d\varphi^2 . \tag{8}$$

We have obtained the metric (8), which satisfy our two conditions that were stated in the previous section. Now we can recalculate the perihelion precession and the deflection of light for (8).

After some tedious calculations, whose details are given in the Appendix II section, the metric in (8) leads to the following precession angle:

$$\delta = \frac{6\pi GM}{ac^2 (1 - e^2)}, \tag{9}$$

where δ is the precession angle, a is the orbit major semi axis, and e is the eccentricity. Notice that δ in (9) is equivalent to the precession predicted by classical general relativity (see [2], p. 197).

However, we cannot find the same agreement between (8) and the Schwarzschild metric, for the deflection of light because (8) gives the deflection of light as (see Appendix III section for detailed calculations):

$$\Delta = \frac{3\pi}{2}\frac{GM}{c^2 R_0}, \tag{10}$$

where Δ is the total deflection and R_0 is the smallest distance of light from the center of the sun. According to classical general relativity (see [2], p. 190), the deflection is

$$\Delta_S = 4\frac{GM}{c^2 R_0}.$$

3) Conclusions

Although, it seems that the main ideas and results of this study conflict with the majority of the ideas of general relativity, the primary disagreement is resulted from the Einstein's limit condition between general relativity and Newtonian gravity. Einstein proposed as a limit condition that (see [2], p. 152)

$$\nabla^2 g_{00} \propto G\rho,$$

where ρ is the mass density. In classical general relativity, Einstein connected metric tensors with only gravitation by this condition. However in this study it is proposed that

$$g_{00} \propto L^2 \tag{11}$$

is the limit condition. L can be the relativistic Lagrangian of any system or interaction, so there is no any constraint for L to belong to any system or any interaction.

As a result of the above facts, in classical general relativity all derived metrics or solutions are based on the Einstein's limit condition. Consequently, all derived metrics, solutions and predictions are some outcomes of this condition. Since the Schwarzschild metric is one of these outcomes, it also contains features of the Einstein's limit condition. Subsequently, black holes, horizons and all other predictions of the Schwarzschild metric are results of the Einstein's limit condition.

However, in this study the metric in (8) is a result of (1) and (11). It contains outcomes of these two requirements and does not lead to the well-known concepts of classical general relativity or the Schwarzschild metric, and actually changes some of them. At this stage, all these points are left as some extraordinary points to be investigated in future studies. But there is a pleasantly clear point: although (8) gives the same perihelion procession amount with classical general relativity, it gives a different deflection amount for light. We know that observations do not exactly confirm the Schwarzschild metric's prediction on the deflection of light. Therefore, the discrepancy between the Schwarzschild metric's prediction and our metrics prediction for the deflection of light is a key factor to test the validity of (8) and consequently to test the validity of (1) and (11).

Appendix I

In this section we derive relativistic Lagrangian for some objects whose rest mass is m_0 and speed is adequately small, and we give some other useful calculations.

Consider a freely moving object whose rest mass is m_0. As a result of reasons that are mentioned in previous sections, the metric for this object will be

$$ds^2 = -c^2 \left(\frac{L}{L_0} \right)^2 dt^2 + dx^2 + dy^2 + dz^2 . \qquad \text{(AI.1)}$$

Dividing (AI.1) by $d\tau$, where τ is the proper time, yields

$$\left(\frac{ds}{d\tau} \right)^2 = -c^2 \left(\frac{L}{L_0} \right)^2 \dot{t}^2 - \overline{v}^2 , \qquad \text{(AI.2)}$$

where the dot denotes $d / d\tau$ and

$$\overline{v}^2 = -\left(\frac{dx}{d\tau} \right)^2 - \left(\frac{dy}{d\tau} \right)^2 - \left(\frac{dz}{d\tau} \right)^2 .$$

We can choose $\left(\frac{ds}{d\tau} \right)^2$ as any constant (say $-b^2$). Thus (AI.2) becomes

$$b^2 = c^2 \left(\frac{L}{L_0} \right)^2 \dot{t}^2 + \overline{v}^2 \qquad \text{(AI.3)}$$

(AI.3) gives the following geodesic equation for t

$$\frac{d}{d\tau} \left[\left(\frac{L}{L_0} \right)^2 \dot{t} \right] = 0 . \qquad \text{(AI.4)}$$

Solution of (AI.4) is

$$\dot{t} = \varepsilon \left(\frac{L}{L_0} \right)^{-2} , \qquad \text{(AI.5)}$$

where ε is the integration constant and can be determined easily. Since (AI.5) is valid for all values of L and $\bar{v}$, for an object at rest, in order to obtain a flat space-time metric, $L = L_0$ (see (AI.2)) and $\dot{t} = \frac{dt}{d\tau} = 1$ can be written, so $\varepsilon = 1$. By substituting (AI.5) into (AI.3) we have

$$b^2 = c^2 \left(\frac{L}{L_0} \right)^{-2} + \bar{v}^2 . \tag{AI.6}$$

(AI.6) is also valid for all values of L and $\bar{v}$. Consider again an object at rest, thus

$$b^2 = c^2 \tag{AI.7}$$

can be found. So (AI.6) can be written as

$$c^2 = c^2 \left(\frac{L}{L_0} \right)^{-2} + \bar{v}^2$$

or

$$L = \frac{L_0}{\sqrt{1 - \bar{v}^2 / c^2}}$$

which is the relativistic Lagrangian and the relativistic kinetic energy of a freely moving object. We can find easily that

$$L_0 = m_0 c^2 ,$$

then

$$L = \frac{m_0 c^2}{\sqrt{1 - \bar{v}^2 / c^2}} . \tag{AI.8}$$

Since (AI.8) is the relativistic Lagrangian of a freely moving object, for an object that is moving in any potential the Lagrangian should be

$$L = \frac{m_0 c^2}{\sqrt{1 - \overline{v}^2 / c^2}} + \frac{1}{2} m_0 v^2 - U \,, \qquad \text{(AI.9)}$$

where (do not confuse $d / d\tau$ with d / dt)

$$v^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$$

If we do not wish to see relativistic effects for slowly moving objects, we can assume that $\overline{v}^2 / c^2 \approx 0$. Thus

$$L \cong m_0 c^2 + \frac{1}{2} m_0 v^2 - U \qquad \text{(AI.10)}$$

can be obtained. Notice that (AI.10) is equivalent to the classical mechanics' Lagrangian, since the additional constant terms in the Lagrangian can always be ignored in classical mechanics. In this case the action function satisfies (6).

If we want to see relativistic effects we assume that $\overline{v}^2 / c^2 \neq 0$ but $\overline{v}^2 / c^2 \ll 1$. This enables us to write $\overline{v} \cong v$ and to expand (AI.9) as (for this case the action function satisfies (7))

$$L \cong m_0 c^2 + \frac{1}{2} m_0 v^2 + \frac{1}{2} m_0 v^2 - U \,,$$

$$L \cong m_0 c^2 + m_0 v^2 - U \,.$$

For a planet around the sun, since

$$\frac{m_0 v^2}{r} = \frac{G M m_0}{r^2} \,,$$

$$L \cong m_0 c^2 + \frac{2GMm_0}{r}$$

can be found.

Appendix II

In this section, we calculate perihelion precession for the metric in (8). Geodesic equations of (8) are:

$$\frac{d\dot{t}}{d\tau} = -2\frac{r_S \dot{t}\dot{r}}{r^2 + r_S r}, \qquad \text{(AII.1)}$$

$$\frac{d\dot{r}}{d\tau} = -c^2 \frac{(r_S + r)r_S}{r^3}\dot{t}^2 + r\dot{\theta}^2 + rSin^2\theta\dot{\varphi}^2, \qquad \text{(AII.2)}$$

$$\frac{d\dot{\theta}}{d\tau} = -2\frac{\dot{r}\dot{\theta}}{r} + Cos\theta Sin\theta\dot{\varphi}^2, \qquad \text{(AII.3)}$$

$$\frac{d\dot{\varphi}}{d\tau} = -2\frac{(\dot{r} + r\dot{\theta}Cot\theta)}{r}\dot{\varphi}, \qquad \text{(AII.4)}$$

where dots denote $d/d\tau$, and τ denotes the proper time.

We can choose axes such that $\theta = \pi/2$, so $\dot{\theta} = 0$. Now we can solve (AII.1) and (AII.4) immediately. (AII.1) gives

$$\dot{t} = \varepsilon\left(1 + \frac{r_S}{r}\right)^{-2}, \qquad \text{(AII.5)}$$

where ε is the integration constant. (AII.4) gives:

$$\dot{\varphi} = \frac{j}{r^2}, \qquad \text{(AII.6)}$$

where j is the integration constant.

Instead of solving (AII.2) we can derive a simpler equation for r. For this we divide (8) by $d\tau^2$, and we choose $\left(ds/d\tau\right)^2 = -1$. So we have

$$-1 = -c^2\left(1+\frac{r_S}{r}\right)^2 \dot{t}^2 + \dot{r}^2 + r^2\dot{\theta}^2 + r^2 Sin^2\theta\dot{\varphi}^2 . \qquad \text{(AII.7)}$$

By substituting (AII.5) and (AII.6) into (AII.7) we get

$$-1 = -c^2\varepsilon^2\left(1+\frac{r_S}{r}\right)^{-2} + \dot{r}^2 + \frac{j^2}{r^2} . \qquad \text{(AII.8)}$$

In order to get a simpler equation, we change $u = 1/r$ and write derivates with respect to φ. (AII.8) becomes

$$-1 = -c^2\varepsilon^2\left(1+r_S u\right)^{-2} + j^2u'^2 + j^2u^2 , \qquad \text{(AII.9)}$$

where prime denotes $d/d\varphi$.

The integration constant ε can be determined by using (AII.9). Since (AII.9) is valid for all r when $r \to \infty$, $u = 1/r \to 0$, and $u' = -r'/r^2 \to 0$. Thus,

$$\varepsilon^2 = 1/c^2$$

can be obtained. Now the metric (AII.9) becomes

$$-1 = -\left(1+r_S u\right)^{-2} + j^2u'^2 + j^2u^2 . \qquad \text{(AII.10)}$$

We differentiate (AII.10) with respect to φ, and we have

$$u'' + u = \frac{r_S}{j^2}\left(1+r_S u\right)^{-3} . \qquad \text{(AII.11)}$$

In order to determine u we need to solve (AII.11). Unfortunately, however, it is a non-linear differential equation and cannot be solved exactly. Fortunately, since $r_S u \ll 1$ for the solar system, $(1+r_S u)^{-3} \approx 1-3r_S u$, can be written. Then (AII.11) can be written as

$$u''+u=\frac{r_S}{j^2}(1-3r_S u),$$

$$u''+\left(1+\frac{3r_S^2}{j^2}\right)u=\frac{r_S}{j^2}. \qquad \text{(AII.12)}$$

Since the path of a planet is investigated, and the path of a planet is an ellipse, j^2 must be such that

$$\frac{r_S}{j^2}=\frac{1}{a(1-e^2)}.$$

Consequently,

$$\frac{r_S^2}{j^2}=\frac{r_S}{a(1-e^2)}.$$

Now taking into consideration that $r_S^2/j^2 \ll 1$, (AII.12) can be solved and u can be found as

$$u \cong \frac{1-eCos\overline{\varphi}}{a(1-e^2)},$$

where

$$\overline{\varphi}=\varphi\left(1+\frac{3r_S}{a(1-e^2)}\right)^{1/2}$$

and $\overline{\varphi}$ is the essential factor to determine perihelion precession. Since $3r_S/a(1-e^2) \ll 1$

$$\left(1+\frac{3r_S}{a(1-e^2)}\right)^{1/2} \approx \left(1+\frac{1}{2}\frac{3r_S}{a(1-e^2)}\right).$$

Thus $\bar{\varphi}$ becomes

$$\bar{\varphi} \cong \varphi\left(1+\frac{3GM}{ac^2(1-e^2)}\right). \tag{AII.13}$$

Consequently, total perihelion precession is (using that for complete orbit $\varphi = 2\pi$)

$$\delta \cong \frac{6\pi GM}{ac^2(1-e^2)}. \tag{AII.14}$$

Appendix III

In this section, we calculate the deflection of light for the metric in (8). Since light, which travels in a straight line, is considered in this case, the right hand side of (AII.12) can be dropped and u can be taken as

$$u = (1/R_0)Cos\bar{\varphi} \tag{AIII.1}$$

where R_0 is the smallest distance of light from the center of the sun, and $\bar{\varphi} = \varphi\left(1+3r_S^2/j^2\right)^{1/2} \cong \varphi\left(1+3r_S^2/2j^2\right)$ again. Naturally, this time the constant j has a different value and in order to determine the deflection of light, j must be found in terms of the known quantities. (AII.10) can be used, noting that φ runs from $-\pi/2$ to $+\pi/2$ and that when $r = R_0$, $\varphi = 0$.

$$\left[-1 = -(1+r_S u)^{-2} + j^2 u'^2 + j^2 u^2\right]_{r=R_0}$$

$$-1 = -\left(1 + \frac{r_S}{R_0}\right)^{-2} + j^2 0^2 + \frac{j^2}{R_0^2}.$$

By assuming that $R_0 \gg r_S$, the last equation can be written as

$$-1 \cong -\left(1 - 2\frac{r_S}{R_0}\right) + \frac{j^2}{R_0^2}.$$

Thus,

$$j^2 \cong -2 r_S R_0. \tag{AIII.2}$$

With the correction, φ runs from $-(\pi/2 + \Delta/2)$ to $(\pi/2 + \Delta/2)$, where Δ is the full angle of the deflection. Setting $u = 0$ ($r \to \infty$) for $\varphi = \pi/2 + \Delta/2$, and treating Δ and r_S / R_0 as small, from (AIII.1) and (AIII.2)

$$Cos\left[\left(\frac{\pi}{2} + \frac{\Delta}{2}\right)\left(1 - \frac{3r_S}{4R_0}\right)\right] = 0,$$

$$Cos\left[\frac{\pi}{2} + \frac{\Delta}{2} - \frac{3\pi r_S}{8R_0} - \frac{3\Delta r_S}{8R_0}\right] = 0. \tag{AIII.3}$$

In (AIII.3) $3\Delta r_S / 8R_0$ is negligible when compared with other terms, so it can be dropped. By expanding (AIII.3),

$$-Sin\left[\frac{\Delta}{2} - \frac{3\pi r_S}{8R_0}\right] \cong 0$$

can be obtained. From the last expression

$$\frac{\Delta}{2} - \frac{3\pi r_S}{8R_0} \cong 0,$$

$$\Delta \cong \frac{3\pi r_S}{4R_0} = \frac{3\pi}{2}\frac{GM}{c^2 R_0}. \tag{AIII.4}$$

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