Covariance of the Schrödinger equation under low velocity boosts.

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It is well-known that Schrödinger wave functions are not covariant under Galilean boosts. To obtain the correct result the boost transformation, $t' = t$ and $\vec{x}' = \vec{x} - \vec{v}t$, must be followed by the phase shift $\delta\phi = \frac{1}{2}mv^2t + m\vec{v}\cdot\vec{r}$. A generally accepted approach is to absorb the phase shift into the Galilean boost, construct the Schrödinger group and claim Galilean invariance of the Schrödinger wave function. Here I address the physical meaning of the phase shift. It is not a coordinate transformation since it depends on the mass of the Schrödinger particle. Consequently, one needs as many Schrödinger groups as there are distinct masses. The phase shift does not follow from Lorentz boost per se in the low velocity limit. Covariance of the non-relativistic quantum mechanical kinetic energy and momentum under pure coordinate transformations can be satisfied only by the boost $t' = t\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{\vec{v}\cdot\vec{r}}{c^2}\right)$ and $\vec{x}' = \vec{x} - \vec{v}t$. Thus proper time and relativity of simultaneity are seen to be the roots of non-relativistic quantum mechanical inertia.

Keywords: Schroedinger, Galilei covariance, Galilei boost.

Introduction

The Schrödinger equation [1],

$$i\partial_t\psi(t,\vec{r}) = -\frac{\nabla^2}{2m}\psi(t,\vec{r}), \tag{1}$$

is not covariant under Galilean boosts. Consider a solution describing a free massive neutral particle with momentum $\vec{p}$ and kinetic energy $\frac{p^2}{2m}$,

$$\psi(t,\vec{r}) = e^{-i\frac{p^2}{2m}t+i\vec{p}\cdot\vec{r}}. \tag{2}$$

A Galilean boost,

$$\begin{aligned} t' &= t, \\ \vec{r}\,' &= \vec{r}-\vec{v}t, \end{aligned} \tag{3}$$

transforms this into

$$\psi(t',\vec{r}\,') = e^{-i(\frac{p^2}{2m}-\vec{p}\cdot\vec{v})t'+i\vec{p}\cdot\vec{r}\,'}. \tag{4}$$

This function has the wrong momentum and kinetic energy, namely $\vec{p}$ and $\frac{p^2}{2m}+\vec{p}\cdot\vec{v}$ instead of $\vec{p}-m\vec{v}$ and $\frac{(\vec{p}-m\vec{v})^2}{2m}$.

An equivalent statement of the problem is that the Schrödinger energy and momentum operators $i\partial_t$ and $-i\vec{\nabla}$ in Galilean relativity transform as

$$\begin{aligned} \partial'_t &= \partial_t+\vec{v}\cdot\vec{\nabla}, \\ \vec{\nabla}\,' &= \vec{\nabla}. \end{aligned} \tag{5}$$

According to the correspondence principle energy and momentum should follow a common transformation law in classical and quantum mechanics. Eq. (5) is at variance with this.

The standard approach [2,3] to this problem is to *redefine* the Galilean transformation by extending it with a phase shift, so that the wave function in the boosted frame is

$$\psi'(t', \vec{r}') = e^{-\frac{1}{2}imv^2t - im\vec{v}\cdot\vec{r}}\psi(t, \vec{r} - \vec{v}t). \tag{6}$$

For the plane wave (2) this leads to the correct result

$$\psi'(t', \vec{r}') = e^{-i\frac{(p-m\vec{v})^2}{2m}t + i(\vec{p}-m\vec{v})\cdot\vec{r}}. \tag{7}$$

From the fact that the phase shift is independent of the momentum and energy of the plane wave it follows that the redefinition (6) correctly transforms any solution of the Schrödinger equation. The redefinition (6) leads to the Schrödinger group, which is actively studied in the literature [4].

The phase shift approach can be criticised on the following grounds. The phase shift depends on the mass of the Schrödinger particle, so that it cannot be a coordinate transformation. The Schrödinger group therefor depends on mass and different groups apply to particles of different mass. The Schrödinger group thus is not a group of coordinate transformations such as the Galilean group and the Poincaré group, which apply to all matter. The phase shift obviously does not follow the Lorentz boost, nor from any other physical principle.

It is now shown that in order to remove the mass dependence and arrive at covariance on the basis of established physical principle, proper time as well as relativity of simultaneity need to be taken taken into account in low velocity boosts. A pure coordinate transformation is now derived that describes a low velocity boost and leads to the correct boosted energy and momentum. Firstly, it is noted that the Schrödinger wave function (2) is not the low velocity limit of a relativistic wave function. This is repaired by including the rest energy into Eq. (1), which becomes

$$i\partial_t\psi(t,\vec{r}) = (-\frac{\nabla^2}{2m} + mc^2)\psi(t,\vec{r}), \tag{8}$$

The equivalent of (2) then becomes

$$\psi(t,\vec{r}) = e^{-imc^2t - i\frac{p^2}{2m}t + i\vec{p}\cdot\vec{r}}. \tag{9}$$

Note that this wave function is still not covariant under Eqs. (3). Secondly, it is noted that in an expansion of the Lorentz boost terms of order c^{-2} contribute terms of order c^0 in the final result if they multiply with the rest frame energy mc^2. These must be retained when taking the limit to infinite c of the transformation, which leads to

$$\begin{aligned} t' &= (1+\frac{1}{2}\frac{v^2}{c^2})t - \frac{\vec{v}\cdot\vec{r}}{c^2}, \\ \vec{r}\,' &= \vec{r} - \vec{v}t. \end{aligned} \tag{10}$$

Further terms of order $\frac{v^2}{c^2}$ do not result in low-velocity effects since they are not involved in a product with the rest energy. Eqs. (10) transform the wavefunction (9) into the correct result

$$\psi(t',\vec{r}\,') = e^{-imc^2t - i\frac{(\vec{p}-m\vec{v})^2}{2m}t + i(\vec{p}-m\vec{v})\cdot\vec{r}}. \tag{11}$$

The inverse boost is obtained by substitution of $-\vec{v}$ into Eqs. (10). A boost followed by its inverse leaves wavefunction (9) invariant up to terms vanishing at least as c^{-2}. Eqs. (10) imply

$$\begin{aligned} \partial_t' &= (1+\frac{1}{2}\frac{v^2}{c^2})\partial_t + \frac{\vec{v}\cdot\vec{\nabla}}{c^2}, \\ \vec{\nabla}\,' &= \vec{\nabla} + \vec{v}\partial_t, \end{aligned} \tag{12}$$

to the same approximation, which is equivalent to Eqs. (10). Thus Eqs. (10) satisfy the correspondence principle.

The two extra terms in Eqs. (10), as compared to Eqs. (3) involve essential aspects of special relativity. The term $\frac{1}{2}\frac{v^2}{c^2}t$, which implies that proper time effects, is required for the covariance of Schrödinger kinetic energy. The term $\frac{\vec{v}\cdot\vec{r}}{c^2}$, which implies that simultaneity of events at different positions is relative, is required for the covariance of Schrödinger momentum.

It is interesting to apply the present approach to the twin system example of Greenberger. Consider a particle of mass m described by Eq. (9) that is observed from a non-inertial coordinate system. In the spirit of Ref. [5], Eqs. (8), it is assumed that the coordinate transformation is

$$\begin{aligned} t' &= (1+\frac{1}{2}\frac{v^2}{c^2})t - \frac{\vec{v}\cdot\vec{r}}{c^2}, \\ \vec{r}\,' &= \vec{r} - \xi(t). \end{aligned} \tag{13}$$

In the case that $\xi(t) \neq 0$ only for $0 < t < t_1$, the wavefunction at $t > t_1$ acquires a phase shift relative to a system with $\xi = 0$ at all times of

$$\phi = \int_0^{t_1} dt \frac{1}{2} m\dot{\xi}^2, \tag{14}$$

which agrees with Greenberger's result. A superposition of two wavefunctions of different energy is not needed here, since the full rest energy is included. As in Greenberger's example, two observers with a different non-inertial history will in general observe a phase difference.

In conclusion, the Galilean boost extended with a mass dependent phase shift leads to a different symmetry group for each value of the mass. This approach clearly fails for a system with particles of different mass. The so called Schrödinger group differs fundamentally from the Galilei group and Poincaré groups, which are valid for all matter. A pure coordinate boost that transforms quantum energy and momentum in the non-relativistic domain correctly is proposed in Eqs. (10) and contains two terms of order $\frac{1}{c^2}$. The first guarantees energy covariance and the second momentum covariance. Thus phenomena of proper time and relativity of simultaneity are seen to be of central importance even in non-relativistic quantum mechanics.

References

[1] Units are chosen such that $\hbar = 1$.

[2] J.M. Lévy-Leblond, Commun. Math. Phys. 6 (1967) 286–311.

[3] See the review paper by H. Brown and P. R. Holland, Am. J. Phys. 67 (1999) 204, and the many references therein.

[4] See for example R. Jackiw and V.P. Nair, Phys. Lett. B480 (2000) 237; C.R. Hagen, Phys. Lett. B539 (2002) 168; P. A. Horvathy, hep-th/0602133.

[5] D. M. Greenberger, Phys. Rev. Lett. 87 (2001) 100405.