

Maxwell's Equations: New Light on Old Problems

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Maxwell's equations possess a certain generic structural property which is well-known, but rarely discussed. By considering this property as primary, we are able to derive the complete mathematical structure of Maxwell's equations described in terms of the orthogonality properties defined between certain spaces of linear operators. But, we find that the classical theory, whilst recovered intact here, is incomplete in the sense that the recovered Maxwell field is irreducibly associated with an additional massive vector field. In the overall context, this massive vector field can only be interpreted as a manifestation of a classical massive photon. One immediate consequence is that the Lorentz force law must be generalized and can be trivially made perfectly Newtonian once the massive vector field is accounted for.

1. Introduction

1.1. Historical Overview

Maxwell's equations, encapsulating as they do over one hundred years of observation and experimentation, arguably represent the ultimate synthesis of the scientific age. For all engineers, and for

some physicists, they are inevitably cast in Heavyside's vectorial form—the so-called Maxwell-Heavyside equations. The concept of *photon* plays no part in this theory—everything is arbitrated by the electromagnetic field in conjunction with the Lorentz force law.

During the first half of the 20th century, Maxwell's equations were given a new compact formulation—the canonical covariant formulation which expresses the electromagnetic field tensor in terms of the four-vector potential. At one level, this last step seemed to be no more than an advance of notation more suited to the requirements of theoretical physics than the Heavyside formulation. However, by the middle of the 20th century, with the development of quantum electrodynamics (qed), it became recognized that the electromagnetic field, when defined in terms of the four-vector potential, is the gauge field which must be introduced to guarantee invariance of a certain action under a local $U(1)$ gauge transformation—and along with this there came a corresponding gauge particle, the *massless photon*. In this way, the insights of mid-20th century theoretical physics were seen to validate and expand the insights of mid-19th century theoretical physics.

Notwithstanding the fact that qed requires photons to be massless and that there is no direct physical evidence that photons are anything other than massless, the idea of the massive photon is a persistent one that refuses to go away. At a fundamental level, for example, it is an implicit requirement of “pilot wave” interpretations of quantum mechanics and, as such, is primarily associated with the names of De Broglie [1,2], who originated the idea for single-particle systems and Bohm [3,4], who conceived it independently and subsequently extended it to multi-particle systems.

Of course, the photon idea plays no explicit role in either interpretation of non-relativistic quantum mechanics but, if the pilot wave interpretation is ever to receive a fully consistent qed-generalization (there has been much preliminary discussion: for

example, see Bohm & Hiley [5], Holland [6], Bell [7] or Cushing *et al.* [8]) then the photon must be conceived as a massive extended particle. It then becomes problematic that there has hitherto been no *independent* theoretical imperative for introducing the idea of the massive photon—thus, if massive photons are required for the theory, then they must be “put in by hand”, usually by constructing some variation of standard electromagnetic theory. A very recent example of this approach is provided by Vigier [9].

The present approach to the idea of the massive photon is distinguished from earlier approaches in the sense that, rather than modifying classical theory with *ad hoc* additions designed to give rise to some variety of the idea, we are able to show how an analysis based on certain generic properties of Maxwell’s equations—rather than on the equations themselves—leads to the unavoidable conclusion that the classical Maxwell field is necessarily and irreducibly associated with a massive vector field (that is, that wherever the Maxwell field exists, then so does the massive vector field and *vice versa*). The irreducible nature of the association leads to the obvious identification of the massive vector field as the classical description of the massive photon.

1.2. Overview of present work

The considerations of this paper were not driven by any attempt to obtain an “improved” electrodynamics, nor to address any hypothetical shortcomings of the classical theory. They were driven, rather, by a spirit of curiosity concerning the general structure of Maxwell’s equations: let us refer to this as *Property A*, defined below:

Property A: The equations of the canonical covariant Maxwell theory can be expressed as identities arising from the mutual orthogonality which exists between certain linear operator spaces. This is shown in detail in §2.1.

Subsequently, by accepting the central position of the Poincaré group within modern physics (*Property B* say), we set about the general problem of how to derive theories which possessed both *Property A* & *Property B*—the expectation was that, by definition, we must recover Maxwell’s equations plus, perhaps, some other things. Maxwell’s equations are indeed recovered intact, *but only in the context of being one-half of a bigger theory*. That is, the dual requirements of *Property A* plus *Property B* lead to the unavoidable conclusion that we *cannot* have the classical Maxwell field in isolation, but that it is irreducibly associated with an additional massive vector field.

1.3. Logical necessity and possible consequences

It is worth emphasizing the logical necessity of the foregoing:

- Maxwell’s equations possess both *Property A* and *Property B*;
- A general search for theories possessing both *Property A* and *Property B* leads to the conclusion that the Maxwell field cannot exist in isolation—it is unavoidably and irreducibly associated with an additional massive vector field; where one is, the other is & *vice versa*.

Thus, the logical situation is that, if we accept the Maxwell field at all, then we must necessarily accept that an additional massive vector field is irreducibly associated with it.

It is this latter property which is of particular interest: specifically, whilst the “photon as particle” can never be recovered from a purely classical theory such as the one considered here, the irreducible association of a massive vector field with the Maxwell field is entirely new. It is also fascinating because it suggests the immediate possibility that the structure of the Lorentz force law might also need generalizing to account for the irreducible presence of this massive vector field. As we shall see in §10, this turns out to be the case and the generalized form can easily be structured so that it becomes fully Newtonian.

1.4. Notation note

We use the convention that (x^1, x^2, x^3) represent the spatial axes and $x^4 \equiv ict$ represents the temporal one with a correspondingly consistent notation for the four-vector current, J_a and the electromagnetic field tensor, F_{ab} .

2. Identities in Canonical Electromagnetic Theory

2.1. Basic observations

When expressed in terms of the field tensor, the microscopic Maxwell's equations in the presence of charge are conventionally written

$$\frac{\partial F_{ai}}{\partial x^i} = \frac{4\pi}{c} J_a, \quad (1)$$

for a conserved current $\mathbf{J} \equiv (\mathbf{j}, ic\rho)$, together with

$$\frac{\partial F_{st}}{\partial x^r} + \frac{\partial F_{tr}}{\partial x^s} + \frac{\partial F_{rs}}{\partial x^t} = 0. \quad (2)$$

It is well known that, when the four-vector potential $\Phi \equiv (\phi_1, \phi_2, \phi_3, \phi_4)$ is introduced and F_{ab} defined according to

$$F_{ab} \equiv \frac{\partial \phi_b}{\partial x^a} - \frac{\partial \phi_a}{\partial x^b}, \quad (3)$$

then (2) becomes identically satisfied. However, because \mathbf{J} in (1) is conserved then, from (1), we have

$$\frac{\partial F_{ai}}{\partial x^i} = \frac{4\pi}{c} J_a \iff \frac{\partial^2 F_{ij}}{\partial x^i \partial x^j} = 0 \quad (4)$$

so that these last two equations are mutually equivalent. But since the second of these equations is also an identity under the definition (3), then we can say that the covariant formulation of

Maxwell's equations can be reduced to a pair of identities, (2) and the second of (4). The physics, of course, comes in when the conserved current, \mathbf{J} , is identified with the flow of charge.

2.2. Interpretation in terms of orthogonal operators

If we now write (3) as

$$F_{ab} \equiv \frac{\partial \phi_b}{\partial x^a} - \frac{\partial \phi_a}{\partial x^b} \equiv P_{ab}^k \phi_k,$$

where the P_{ab}^k , $k=1..4$ are linear differential operators, then the identities (2) and (4) can be formally expressed as

$$\frac{\partial F_{st}}{\partial x^r} + \frac{\partial F_{tr}}{\partial x^s} + \frac{\partial F_{rs}}{\partial x^t} \equiv R_{rst}^{ij} P_{ij}^k \phi_k = 0,$$

$$\frac{\partial^2 F_{ij}}{\partial x^i \partial x^j} \equiv Q^{ij} P_{ij}^k \phi_k = 0 \quad (5)$$

respectively, for linear differential operators Q^{ab} and R_{rst}^{ab} . Since an entirely *arbitrary* definition of $(\phi_1, \phi_2, \phi_3, \phi_4)$ satisfies (5), it follows that $Q^{ij} P_{ij}^k \equiv 0$ and $R_{rst}^{ij} P_{ij}^k \equiv 0$; that is, the canonical covariant form of Maxwell's equations can be considered based on algebraic orthogonality properties between sets of linear differential operators.

In the following, we use this insight into the nature of Maxwell's equations to write down a Lagrangian formulation of the most general theory possible that is defined over a two-index field and which leads to equations which are essentially identities in the above sense for Maxwell's equations.

3. A Lagrangian Density

We argue, in Appendix A, that the Lagrangian density which leads to the required theory must have the general structure:

$$\begin{aligned}
L = & \alpha_0 \frac{\partial \Psi_{ij}}{\partial x^k} \frac{\partial \Psi_{ji}}{\partial x^k} + \alpha_1 \frac{\partial \Psi_{ij}}{\partial x^k} \frac{\partial \Psi_{ij}}{\partial x^k} \\
& + \beta_0 \frac{\partial \Psi_{ik}}{\partial x^i} \frac{\partial \Psi_{kj}}{\partial x^j} + \beta_1 \left(\frac{\partial \Psi_{ik}}{\partial x^i} \frac{\partial \Psi_{jk}}{\partial x^j} + \frac{\partial \Psi_{ki}}{\partial x^i} \frac{\partial \Psi_{kj}}{\partial x^j} \right) \\
& + \gamma_0 \frac{\partial \Psi_{ik}}{\partial x^j} \frac{\partial \Psi_{kj}}{\partial x^i} + \gamma_1 \left(\frac{\partial \Psi_{ik}}{\partial x^j} \frac{\partial \Psi_{jk}}{\partial x^i} + \frac{\partial \Psi_{ki}}{\partial x^j} \frac{\partial \Psi_{kj}}{\partial x^i} \right),
\end{aligned}$$

where $(\alpha_0, \alpha_1, \beta_0, \beta_1, \gamma_0, \gamma_1)$ are arbitrary constants. However, it turns out that this density contains a large amount of redundancy; specifically, all the independent structure is retained if only one of (α_0, α_1) is non-zero and only one of $(\beta_0, \beta_1, \gamma_0, \gamma_1)$ is non-zero. Consequently, the working density can be assumed to be

$$L = -\lambda \frac{\partial \Psi_{ij}}{\partial x^k} \frac{\partial \Psi_{ji}}{\partial x^k} + \frac{\partial \Psi_{ik}}{\partial x^i} \frac{\partial \Psi_{kj}}{\partial x^j}, \quad (6)$$

for free parameter $-\lambda$; the minus has been used for later convenience. From this, the corresponding Euler-Lagrange equations can be found as

$$-\lambda^2 \Psi_{ab} + \frac{\partial}{\partial x^i} \left[\frac{\partial \Psi_{ai}}{\partial x^b} + \frac{\partial \Psi_{ib}}{\partial x^a} \right] = 0, \quad \square \equiv \frac{\partial^2}{\partial x^i \partial x^i}, \quad (7)$$

which is a system of partial differential equations for Ψ_{ab} . Since no assumptions have been made concerning the structure of Ψ_{ab} , $a, b = 1..4$, it can be assumed to contain sixteen degrees of freedom; suppose we represent these degrees of freedom as sixteen sufficiently differentiable functions, $\alpha_k(\mathbf{x}, ct)$, $k = 1..16$ which are, as yet, undetermined and then write

$$\Psi_{ab}(\mathbf{x}, ct) = \sum_{k=1}^{16} U_{ab}^k \alpha_k(\mathbf{x}, ct), \quad (8)$$

where, by analogy with $F_{ab}^k \equiv P_{ab}^k \phi_k$ defined in the previous section and in anticipation of the final result, the two-index objects U_{ab}^k are to be treated as undetermined linear differential operators. Substitution of (8) into (7) gives

$$\sum_{k=1}^{16} \left\{ -\lambda^2 U_{ab}^k + \frac{\partial}{\partial x^i} \left[\frac{\partial U_{ai}^k}{\partial x^b} + \frac{\partial U_{ib}^k}{\partial x^a} \right] \right\} \alpha_k(\mathbf{x}, ct) = 0 \quad (21)$$

where the object $\{\dots\}$ is to be considered as a linear differential operator acting on $\alpha_k(\mathbf{x}, ct)$. We now consider ways of satisfying this equation *identically*: Since λ is an undetermined parameter and the U_{ab}^k are undetermined differential operators, then they can be chosen to satisfy the manifestly Poincaré-invariant relations

$$\frac{\partial}{\partial x^i} \left[\frac{\partial U_{ai}^k}{\partial x^b} + \frac{\partial U_{ib}^k}{\partial x^a} \right] = \lambda^2 U_{ab}^k, \quad k = 1..16. \quad (22)$$

Now define the notation $X_a \equiv \partial/\partial x^a$, $a = 1..4$, $\mathbf{U}^k \equiv (U_{11}^k, U_{12}^k, U_{13}^k, \dots, U_{44}^k)^T$ and sixteen symmetric matrices σ_{mn} , $m, n = 1..4$, each of dimension 16×16 , according to (50) in Appendix B; then (22) can be written as

$$\sigma_{ij} X_i X_j \mathbf{U}^k = \lambda^2 \mathbf{U}^k, \quad k = 1..16 \quad (11)$$

where summation is assumed over i and j . Noting that the eigenvalues of $\sigma_{ij} X_i X_j$ (treated as an algebraic matrix) must be simple multiples of the d'Alembertian, $\square \equiv X_i X_i$, then (11) is seen to have the formal structure of an algebraic eigenvalue problem. Consequently, non-trivial solutions for \mathbf{U}^k can only exist when λ^2 is an eigenvalue of $\sigma_{ij} X_i X_j$; in this case, \mathbf{U}^k is the corresponding eigenvector and must have the structure of a column of differential operators.

Finally, we note how the symmetry of the matrices, σ_{ij} means that, if \mathbf{U}^m and \mathbf{U}^n are eigenvectors corresponding to distinct eigenvalues, then $U_{ij}^m U_{ij}^n = 0$, where summation over i and j is

implied. It is these orthogonality relations which give rise, amongst other things, to the classical equations of electrodynamics.

4. The Eigensystem

Using (22), the eigensystem (11) can be written as

$$X_a X_i U_{ib}^k + X_b X_i U_{ai}^k = \lambda^2 U_{ab}^k, \quad k = 1 \dots 16,$$

and for which we find only five distinct eigenvalues corresponding to $\lambda = 2, 1, 1, 0$ and 0 respectively. The corresponding eigenspaces, which have dimensions one, three, three, three and six respectively, are denoted as $(R)_{sy,1}$, $(R)_{sk,3}$, $(R)_{sy,3}$, $(G)_{sk,3}$ and $(G)_{sy,6}$. We shall show that:

- classical electromagnetism arises from $(R)_{sk,3}$;
- the electromagnetic dual arises from $(G)_{sk,3}$;
- the equations of electromagnetism arise from the orthogonalities $(R)_{sy,1} \perp (R)_{sk,3}$ and $(G)_{sk,3} \perp (R)_{sk,3}$;
- $(R)_{sy,3}$ gives rise to a non-zero mass vector field which is irreducibly associated with the electromagnetic field. We shall argue that this vector field can only be sensibly interpreted as a classical representation of the massive photon.

Only the eigenspace $(G)_{sy,6}$ plays no obvious role in the present discussion. The eigenspaces are described as follows:

4.1. Eigenspace $(R)_{sy,1}$, $\lambda_1 = 2$

$(R)_{sy,1}$ is a one-dimensional subspace of eigenvectors associated with the eigenvalue $\lambda_1 = 2$ and the subspace is defined by the single operator

$$U_{ab}^1 = X_a X_b \quad (12)$$

which is *symmetric* with respect to the indices (a,b) . Consequently, when $\lambda = 2$ in (7), the solution (8) becomes

$$\Psi_{ab} = U_{ab}^1 \alpha_1(\mathbf{x}, ct),$$

so that Ψ_{ab} is defined over a scalar field.

4.2. Eigenspace $(R)_{sk,3}$, $\lambda_2 = 1$

$(R)_{sk,3}$ is a three-dimensional subspace of eigenvectors associated with the eigenvalue $\lambda_2 = 1$ and a basis for the subspace is given by

$$U_{ab}^k = (X_a \delta_{rb} - X_b \delta_{ra}), \quad k = 2, 3, 4 \quad (13)$$

where, for $k = (2, 3, 4)$ then r takes any three distinct values from the set $(1, 2, 3, 4)$; for example, $r = (1, 2, 3)$; these eigenvectors are *skew-symmetric* with respect to the indices (a, b) . Consequently, when $\lambda = 1$ in (7), the solution (8) corresponding to $(R)_{sk,3}$ becomes

$$\Psi_{ab} = \sum_{k=2}^4 U_{ab}^k \alpha_k(\mathbf{x}, ct)$$

so that Ψ_{ab} is defined over a vector field.

4.3. Eigenspace $(R)_{sy,3}$, $\lambda_3 = 1$

$(R)_{sy,3}$ is a three-dimensional subspace of eigenvectors associated with the eigenvalue $\lambda_3 = 1$ and a basis for the subspace is given by

$$U_{ab}^k = X_a (X_r \delta_{sb} - X_s \delta_{rb}) + X_b (X_r \delta_{sa} - X_s \delta_{ra}), \quad k = 5, 6, 7 \quad (14)$$

where for $k = (5, 6, 7)$, then (r, s) is three distinct pairs chosen from $(1, 2, 3, 4)$. The basis is most conveniently chosen by picking any one of the four digits and pairing it with the remaining three: for example, $(r, s) = (1, 4), (2, 4), (3, 4)$. These eigenvectors are *symmetric* with respect to the indices (a, b) . Consequently, when $\lambda = 1$ in (7), the solution (8) corresponding to $(R)_{sy,3}$ becomes

$$\Psi_{ab} = \sum_{k=5}^7 U_{ab}^k \alpha_k(\mathbf{x}, ct)$$

so that Ψ_{ab} is defined over a vector field.

4.4. Eigenspace $(G)_{sk,3}$, $\lambda_4 = 0$

$(G)_{sk,3}$ is a three-dimensional subspace of eigenvectors associated with the eigenvalue $\lambda_4 = 0$ and a basis for the subspace is given by

$$U_{ab}^k = \frac{X_r X_s X_t}{X_a X_b} \left((\delta_{ra} - \delta_{sa})(\delta_{sb} - \delta_{tb}) - (\delta_{rb} - \delta_{sb})(\delta_{sa} - \delta_{ta}) \right); \quad k = 8, 9, 10 \quad (15)$$

where typically, for $k = (8, 9, 10)$ then $(r, s, t) = (2, 3, 4), (1, 3, 4), (1, 2, 4)$; these eigenvectors are *skew-symmetric* with respect to the indices (a, b) . Consequently, when $\lambda = 0$ in (7), the solution (8) corresponding to $(G)_{sk,3}$ becomes

$$\Psi_{ab} = \sum_{k=8}^{10} U_{ab}^k \alpha_k(\mathbf{x}, ct)$$

so that Ψ_{ab} is defined over a vector field.

4.5. Eigenspace $(G)_{sy,6}$, $\lambda_5 = 0$

$(G)_{sy,6}$ is a six-dimensional subspace of eigenvectors associated with the eigenvalue $\lambda_5 = 0$ and a basis for the subspace is given by

$$U_{ab}^k = (X_r \delta_{sa} - X_s \delta_{ra})(X_r \delta_{sb} - X_s \delta_{rb}); \quad k = 11 \dots 16 \quad (16)$$

where, typically, for $k = 11 \dots 16$ then $(r, s) = (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)$ and δ_{ab} is the 4×4 unit matrix; these eigenvectors are *symmetric* with respect to the indices (a, b) . Consequently, when $\lambda = 0$ in (7), the solution (8) corresponding to $(G)_{sy,6}$ becomes

$$\Psi_{ab} = \sum_{k=11}^{16} U_{ab}^k \alpha_k(\mathbf{x}, ct).$$

As we have already noted, and as we shall see in the following, $(G)_{sy,6}$ is the only solution of (22) which does not play

any obvious part in the electromagnetic theory being discussed here.

5. The Electromagnetic Field From $(R)_{sk,3}$

5.1. The Orthogonality Relationships

In this section we show that Maxwell's equations arise naturally as a direct consequence of the orthogonality relations

$$U_{ij}^E U_{ij}^D \equiv 0, \quad \mathbf{U}^E \in (R)_{sy,1}, \quad \mathbf{U}^D \in (R)_{sk,3} \quad (17)$$

$$U_{ij}^B U_{ij}^D \equiv 0, \quad \mathbf{U}^B \in (G)_{sk,3}, \quad \mathbf{U}^D \in (R)_{sk,3} \quad (18)$$

given at (39) in Appendix B and where $(R)_{sy,1}$, $(G)_{sk,3}$ and $(R)_{sk,3}$ are defined at (12), (15) and (13) respectively.

The most general tensor which can be formed from $(R)_{sk,3}$ is given by

$$F_{ab} = \sum_{k=2}^4 U_{ab}^k \alpha_k(\mathbf{x}, ct), \quad (19)$$

$$U_{ab}^k \equiv (X_a \delta_{rb} - X_b \delta_{ra}); \quad k = 2, 3, 4$$

where r takes any three distinct values from $(1, 2, 3, 4)$ and where, because of the skew-symmetry of U_{ab}^k , then F_{ab} is also skew-symmetric. Since $(R)_{sy,1}$ consists of the single operator $X_a X_b$, then (17) with (19) implies

$$X_i X_j F_{ij} \equiv \frac{\partial^2 F_{ij}}{\partial x^i \partial x^j} = 0, \quad (20)$$

from which it immediately follows

$$\frac{\partial F_{ai}}{\partial x^i} = J_a, \quad \text{where} \quad \frac{\partial J_i}{\partial x^i} = 0 \quad (21)$$

for some conserved current \mathbf{J} . Similarly, denoting the elements of $(G)_{sk,3}$ by Δ_{rst}^{ab} , the relation (6) together with (7) gives directly

$$\frac{1}{2} \Delta_{rst}^{ij} F_{ij} \equiv \frac{\partial F_{st}}{\partial x^r} + \frac{\partial F_{tr}}{\partial x^s} + \frac{\partial F_{rs}}{\partial x^t} = 0. \quad (22)$$

If \mathbf{J} in (21) is interpreted as the 4-current density, then (21) and (22) are Maxwell's equations for the *electromagnetic field tensor*, F_{ab} . That is, Maxwell's equations are seen to arise as a direct consequence of the orthogonality between the invariant subspaces $(R)_{sk,3}$, $(R)_{sy,1}$ and $(G)_{sk,3}$. Consequently, in the form of (8) and (22), they impose *no* constraints (beyond differentiability) on the three-vector $\mathbf{A} \equiv (\alpha_{11}, \alpha_{12}, \alpha_{13})$ —this vector field can have arbitrary structure. Since (21) is in a one-to-one relationship with the identity (20) it must also be an identity and *not* a field equation as it is commonly interpreted. The practical use of (21), of course, arises from the identification of \mathbf{J} with a measurable quantity—the four-vector current—which then allows the computations of the fields.

6. Recovery of the canonical four-vector formalism

Although one of (X_1, X_2, X_3, X_4) refers to the temporal axis and three refer to the spatial axes, specific associations of the indices with particular axes have not yet been made. In the following, we show how a simple transformation of $\mathbf{A} \equiv (\alpha_2, \alpha_3, \alpha_4)$:-

- allows the identification of the temporal axis;
- identifies $\mathbf{A} \equiv (\alpha_2, \alpha_3, \alpha_4)$ as the classical magnetic vector potential;
- reduces the presented formalism directly to the canonical formalism.

The expression (7) for the field tensor, F_{ab} , is unconventional insofar as it derives directly from a manifestly covariant treatment

but is not expressed in terms of the usual four-vector potential; instead, it is expressed in terms of an uninterpreted three-vector $\mathbf{A} \equiv (\alpha_2, \alpha_3, \alpha_4)$. In the following, by showing how F_{ab} , defined at (19), can be transformed into the conventional four-vector formalism, we are able to identify \mathbf{A} with the magnetic vector potential of the classical theory whilst, at the same time, identifying the temporal axis. We begin by noting, from (19), that

$$U_{ab}^k \equiv (X_a \delta_{rb} - X_b \delta_{ra}), k = 2, 3, 4$$

where r takes any three values from $(1, 2, 3, 4)$. It is quite obvious that the action of picking three from four here has the effect of making the omitted integer special in some sense which is not yet immediately clear. It transpires that this process effectively associates the omitted index with the temporal axis and the three chosen indices with the spatial axes. For convenience, we begin by defining the object P_{ab}^r according to

$$P_{ab}^r \equiv (X_a \delta_{rb} - X_b \delta_{ra}), r = 1..4, \quad (23)$$

and similarly define the notation that any operand of P_{ab}^r is denoted by $A_r, r = 1..4$. In terms of this notation and after choosing (l, m, n) as any three from $(1, 2, 3, 4)$, then (19) becomes

$$F_{ab} = P_{ab}^l A_l + P_{ab}^m A_m + P_{ab}^n A_n. \quad (24)$$

Additionally, from (23), we readily obtain the identity

$$P_{ab}^r X_r \equiv (X_a \delta_{rb} - X_b \delta_{ra}) X_r \equiv (X_a X_b - X_b X_a)$$

so that, under the assumption that the order of differentiation never matters, we can write the identity

$$P_{ab}^1 X_1 + P_{ab}^2 X_2 + P_{ab}^3 X_3 + P_{ab}^4 X_4 \equiv 0. \quad (25)$$

and, using this, define the three-vector $\mathbf{A}' \equiv (A'_l, A'_m, A'_n)$ according to

$$A'_r = A_r + \frac{X_r}{X_q} A'_q, \quad r = l, m, n \quad (26)$$

where, since $X_q \equiv \partial/\partial x^q$, then $1/X_q$ indicates integration with respect to x^q . Equation (24) can now be rewritten as

$$F_{ab} = P_{ab}^l \left(A'_l - \frac{X_l}{X_q} A'_q \right) + P_{ab}^m \left(A'_m - \frac{X_m}{X_q} A'_q \right) + P_{ab}^n \left(A'_n - \frac{X_n}{X_q} A'_q \right).$$

Using the identity (25) then this last equation can be written as

$$\begin{aligned} F_{ab} &= P_{ab}^l A'_l + P_{ab}^m A'_m + P_{ab}^n A'_n + P_{ab}^q A'_q \\ &= P_{ab}^1 A'_1 + P_{ab}^2 A'_2 + P_{ab}^3 A'_3 + P_{ab}^4 A'_4 \end{aligned} \quad (27)$$

which, with (23), is easily shown to reduce to

$$F_{ab} \equiv X_a A'_b - X_b A'_a.$$

This latter expression is simply the standard form of F_{ab} in terms of the four-vector potential (A'_1, A'_2, A'_3, A'_4) . It is now obvious that $\mathbf{A} \equiv (A_l, A_m, A_n)$, used at (1) to define \mathbf{A}' , is simply the magnetic vector potential of the classical theory and that the arbitrary scalar, A'_q , introduced at (26) is just the scalar potential usually associated with the electric field. This, in turn, implies that the index q is necessarily associated with the temporal axis and the indices (l, m, n) are necessarily associated with the spatial axes.

To summarize, the new formalism based upon $(R)_{sk,3}$ is expressed *entirely* in terms of the classical magnetic vector potential and the canonical formalism is recovered when an arbitrary scalar field is introduced via a certain linear transformation. We discuss the implications of this circumstance in the concluding section.

6.1. The explicit representation of F_{ab} in terms of \mathbf{E} and \mathbf{B} .

In general, we have, from (19),

$$F_{ab} = \sum_{k=2}^4 U_{ab}^k \alpha_k(\mathbf{x}, ct),$$

$$U_{ab}^k \equiv (X_a \delta_{rb} - X_b \delta_{ra}), \quad k = 2, 3, 4,$$

where r is chosen as any three of $(1, 2, 3, 4)$. For convenience, we choose the basis $r \equiv (l, m, n) = (1, 2, 3)$ as $k = (2, 3, 4)$ so that, by the considerations of §6, the indices $r = (1, 2, 3)$ are associated with the spatial axes, the index $r = 4$ is associated with the temporal axis and $\mathbf{A} \equiv (\alpha_2, \alpha_3, \alpha_4)$ is identified as the magnetic vector potential. For the magnetic and electric fields, using the standard notation $\mathbf{B} \equiv (F_{23}, F_{31}, F_{12})$ and $-i\mathbf{E} \equiv (F_{14}, F_{24}, F_{34})$, we find

$$\begin{aligned} \mathbf{B} &= (X_2 A_3 - X_3 A_2, X_3 A_1 - X_1 A_3, X_1 A_2 - X_2 A_1), \\ -i\mathbf{E} &= (-X_4 A_1, -X_4 A_2, -X_4 A_3). \end{aligned}$$

Thus, we see how, whilst the magnetic field takes its standard form in terms of the magnetic vector potential, the form of the electric field in the covariant $(R)_{sk,3}$ formalism differs from the conventional structure in the absence of the scalar potential component.

7. The Dual Field of Electrodynamics From

$(G)_{sk,3}$

The most general tensor generated by $(G)_{sk,3}$ is given by

$$G_{ab} = \sum_8^{10} U_{ab}^k \alpha_k(\mathbf{x}, ct) \quad (28)$$

$$U_{ab}^k = \frac{X_r X_s X_t}{X_a X_b} \left((\delta_{ra} - \delta_{sa})(\delta_{sb} - \delta_{tb}) - (\delta_{rb} - \delta_{sb})(\delta_{sa} - \delta_{ta}) \right); \quad k = 8, 9, 10$$

where, typically, for $k = (8, 9, 10)$ then $(r, s, t) = (2, 3, 4), (1, 3, 4), (1, 2, 4)$. If, for the sake of convenience, we define $(\alpha_8, \alpha_9, \alpha_{10}) \equiv (A_1, A_2, A_3)$, and use the given basis for (r, s, t) , then it is easily found that

$$G_{ab} = \begin{pmatrix} 0 & -X_4 A_3 & X_4 A_2 & X_2 A_3 - X_3 A_2 \\ X_4 A_3 & 0 & -X_4 A_1 & X_3 A_1 - X_1 A_3 \\ -X_4 A_2 & X_4 A_1 & 0 & X_1 A_2 - X_2 A_1 \\ X_3 A_2 - X_2 A_3 & X_1 A_3 - X_1 A_2 & X_2 A_1 - X_1 A_2 & 0 \end{pmatrix}$$

A consideration of this skew-symmetric object soon shows that it is no more than a re-ordering of the terms of the electromagnetic field tensor—which suggests an electrodynamic interpretation of G_{ab} . In fact, it is easily shown that $G_{ab} = \varepsilon_{abmn} F_{mn}$ where ε_{abmn} is the Levi-Civita permutation tensor. Thus, G_{ab} is the dual of F_{ab} .

8. Wave Structures Supported by the Magnetic Vector Potential

It is shown that, according to the generalized electrodynamics, wavy solutions for the magnetic vector potential are composed of two distinct kinds of wave: the first kind is a propagating transverse wave, whilst the second kind, which is novel, is a stationary longitudinal wave. It is shown that the propagating transverse component corresponds identically to those solutions which arise in the conventional formalism when the Coulomb gauge is used. The stationary longitudinal component has no counterpart in the conventional formalism.

Using the notation $\mathbf{A} \equiv (A_1, A_2, A_3) \equiv (\alpha_2, \alpha_3, \alpha_4)$ in (19) and the basis $r = (1, 2, 3)$, then (21) can be written as

$$\sum_{r=1}^3 X_i (X_a \delta_{ri} - X_i \delta_{ra}) A_r = J^a$$

which—upon remembering $X_a \equiv \partial/\partial x^a$ —can be written as the two equations

$${}^2 \mathbf{A} - \nabla(\nabla \cdot \mathbf{A}) = -\mathbf{J} \quad (29)$$

$$\frac{\partial}{\partial x^4}(\nabla \cdot \mathbf{A}) = J_4.$$

Given \mathbf{J} and hence \mathbf{A} via (4), the second of these equations provides a *definition* of J^4 . Consider now, a wave given by

$$\mathbf{A}_{wave} = \mathbf{A}_0 \exp(i\mathbf{n} \cdot \mathbf{x}),$$

where $\mathbf{n} \equiv (n_1, n_2, n_3, n_4)$, $\mathbf{x} \equiv (x^1, x^2, x^3, x^4)$ and \mathbf{A}_0 is a constant three-vector. The requirement that \mathbf{A}_{wave} satisfies (4) with $\mathbf{J} = 0$ leads to the system of equations

$$(\mathbf{n} \cdot \mathbf{n})\mathbf{A}_0 = (\hat{\mathbf{n}} \cdot \mathbf{A}_0)\hat{\mathbf{n}} \quad (30)$$

where $\hat{\mathbf{n}} \equiv (n_1, n_2, n_3)$ and, from this, we can form the scalar equation

$$(\mathbf{n} \cdot \mathbf{n})(\hat{\mathbf{n}} \cdot \mathbf{A}_0) = (\hat{\mathbf{n}} \cdot \mathbf{A}_0)(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}). \quad (31)$$

This latter equation has two possible solutions which, together, form a basis for the homogeneous solutions of (29):

Case 1: The Transverse Wave: $\hat{\mathbf{n}} \cdot \mathbf{A}_0 = 0$

In this case, (30) only has a non-trivial solution if $\mathbf{n} \cdot \mathbf{n} = 0$. Consequently, this solution is given by

$$\mathbf{A}_T = \mathbf{A}_0 \exp(i\mathbf{n} \cdot \mathbf{x}), \quad \mathbf{n} \cdot \mathbf{n} = 0, \quad \hat{\mathbf{n}} \cdot \mathbf{A}_0 = 0, \quad (32)$$

which corresponds to a *transverse* wave propagating with speed c . Since, as noted in the previous section, there is no electric scalar

potential in $(R)_{sk,3}$ and since $\hat{\mathbf{n}} \cdot \mathbf{A}_0 = 0 \rightarrow \nabla \cdot \mathbf{A}_T = 0$, then this component of the general solution of (4) corresponds *exactly* to those solutions which arise from the conventional formalism when the Coulomb gauge is chosen.

Case 2: The Longitudinal Wave: $\hat{\mathbf{n}} \cdot \mathbf{A}_0 \neq 0$

In this case, (31) gives $\mathbf{n} \cdot \mathbf{n} = \hat{\mathbf{n}} \cdot \hat{\mathbf{n}}$ and this can only be true if $n_4 = 0$. From (30) we now get the equation

$$(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}})\mathbf{A}_0 = (\hat{\mathbf{n}} \cdot \mathbf{A}_0)\hat{\mathbf{n}}$$

which is easily seen to have the solution $\mathbf{A}_0 = \alpha \hat{\mathbf{n}}$ for arbitrary α . To summarize, this solution is given by

$$\mathbf{A}_L = \alpha \hat{\mathbf{n}} \exp(i\hat{\mathbf{n}} \cdot \hat{\mathbf{x}}) \quad (33)$$

where $\hat{\mathbf{x}} = (x^1, x^2, x^3)$ and this corresponds to a *longitudinal* stationary wave. This wave is easily shown to give $\mathbf{E} = \mathbf{B} = 0$, so that a non-trivial magnetic vector potential can be associated with a zero electromagnetic field.

To summarize, we arrive at the conclusion that the magnetic vector potential supports two kinds of waves in free space: a propagating transverse wave (which corresponds exactly to the Coulomb gauge solutions of the conventional formalism) and a stationary longitudinal wave (which has no counterpart in the conventional formalism) so that the general wavy solution to the homogeneous form of (29) is given by

$$\mathbf{A}_{wave} = \mathbf{A}_T(\mathbf{x}, ct) + \mathbf{A}_L(\mathbf{x}),$$

where \mathbf{A}_T is the transverse wave propagating with speed c and \mathbf{A}_L is the stationary longitudinal wave. The component \mathbf{A}_T gives rise to propagating transverse electromagnetic fields and the general phenomenology that, when such a field is created, any charged particle anywhere will eventually feel its effect. The component \mathbf{A}_L gives rise to a *zero* electromagnetic field ($\mathbf{E} = \mathbf{B} = 0$) and so no electromagnetic effect at all is propagated; since the stationary wave cannot pass through an arbitrarily placed

charged particle then the only way an effect can be observed is that the charged particle must pass through the stationary wave.

8.1. A Material Vacuum?

It has been shown how, according to the generalized electrodynamics, the magnetic vector potential supports a stationary longitudinal wave, as well as the conventional propagating transverse wave. The classical theory has managed to assimilate the idea of an electrodynamic wave propagating in the absence of any supporting medium—if only because there is, at least, a sense of *something* travelling and things can be conceived as travelling through empty space. However, the idea of stationary longitudinal waves in the absence of any kind of supporting medium presents an entirely new level of incomprehensibility since, now, *nothing* is travelling anywhere. In Appendix D, a possible solution to this perceived problem is obtained by showing how (29) supports a non-wavy $\mathbf{J} = 0$ solution which has a ready interpretation as a classical description of a fluctuating material vacuum. The magnetic vector potential waves discussed in §8 can then be interpreted as *disturbances* of this material vacuum.

9. A Non-Zero Mass Photon From $(R)_{sy,3}$

In this section, we show that $(R)_{sy,3}$ implies the existence of a massive vector field constructed from the elements of $(R)_{sk,3}$, the electromagnetic field. By showing that this vector field is *irreducibly* associated with the electromagnetic field, we are led to the obvious interpretation that this massive vector field is a classical representation of the massive photon.

9.1. The Massive Vector Field

The basis for $(R)_{sy,3}$ is given at (14) and the most general field which can be formed from the operators lying in this subspace is given by

$$V_{ab} = \sum_{k=5}^7 U_{ab}^k \alpha_k(\mathbf{x}). \quad (34)$$

If we define

$$P_{rs}^t = X_r \delta_{st} - X_s \delta_{rt} \quad (35)$$

then the basis for $(R)_{sy,3}$, given above, can be written as

$$U_{ab}^k = (X_a P_{rs}^b + X_b P_{rs}^a); \quad k = 5, 6, 7, \quad (36)$$

where for $k = (5, 6, 7)$, then (r, s) is chosen as in §4.3. With this notation and defining

$$V_a = \sum_{k=5}^7 P_{rs}^a \alpha_k(\mathbf{x}, ct), \quad V_b = \sum_{k=5}^7 P_{rs}^b \alpha_k(\mathbf{x}, ct), \quad (37)$$

then (21) can be expressed as

$$V_{ab} = (X_a V_b + X_b V_a). \quad (38)$$

Since the single element of $(R)_{sy,1}$ is orthogonal to every element of $(R)_{sy,3}$ and since V^{ab} is the most general field which can be formed by the operators lying within this latter subspace acting over a vector field, then operating $(R)_{sy,1}$ onto (34) gives immediately

$$X_i X_j V_{ij} \equiv \frac{\partial^2 V_{ij}}{\partial x^i \partial x^j} = 0,$$

from which it immediately follows

$$\frac{\partial V_{aj}}{\partial x^j} = J_a \quad \text{where} \quad \frac{\partial J_i}{\partial x^i} = 0,$$

for some unspecified current \mathbf{J} . Using (38) this latter equation can be expressed as

$$\frac{\partial}{\partial x^j} \left(\frac{\partial V_j}{\partial x^a} + \frac{\partial V_a}{\partial x^j} \right) = J_a. \quad (39)$$

However, since, from (35), $X_i P_{rs}^i \equiv 0$ then, from the definition of V_a at (37) and the definition of P_{rs}^i at (35), we can easily see that

$$X_j V_j \equiv \frac{\partial V_j}{\partial x^j} = 0, \quad (40)$$

so that (39) becomes

$$\square^2 V_a = J_a. \quad (41)$$

However, since $\partial J_i / \partial x^i = 0$ and $\partial V_i / \partial x^i = 0$, we can write $J_a = mV_a + J_a^0$ for some constant m and conserved current J_a^0 ; finally, therefore, (41) can be written as

$$\square^2 V_a = mV_a + J_a^0. \quad (42)$$

This latter equation implies that $(R)_{sy,3}$ is associated with a massive vector field.

9.2. Does the vector field represent a classical photon?

The first thing to notice, as reference to (23) shows, is that the operator P_{rs}^i defined at (35) and in terms of which the vector field is defined at (37), is the fundamental operator of $(R)_{sk,3}$ —the operator space which acts over the magnetic vector potential to generate the electromagnetic field. Specifically, for the electromagnetic field tensor, we have

$$F_{ab} = \sum_{k=2}^4 P_{ab}^r \alpha_k(\mathbf{x}, ct) \quad (43)$$

where r varies with k whilst, for the massive vector field, we have

$$V_a = \sum_{k=5}^7 P_{rs}^a \alpha_k(\mathbf{x}, ct) \quad (44)$$

where (r, s) varies with k as in §4.3. Now form the inner product of V_a with the four components $A'_i, i=1..4$ defined at (26), to obtain

$$V_i A'_i = \sum_{k=5}^7 P_{rs}^i A'_i \alpha_k(\mathbf{x}, ct).$$

But, by (27), $F_{rs} = P_{rs}^i A'_i$ and so this latter equation becomes

$$V_i A'_i = \sum_{k=5}^7 F_{rs} \alpha_k(\mathbf{x}, ct). \quad (45)$$

First choice of basis: If we fix the basis of (36) by choosing $(r, s) = (1, 4), (2, 4), (3, 4)$ and use the notation $\mathbf{a} \equiv (\alpha_5, \alpha_6, \alpha_7)$ and $(F_{14}, F_{24}, F_{34}) = -i(E_1, E_2, E_3)$ then (7) gives

$$V_i A'_i = -i\mathbf{E} \cdot \mathbf{a}. \quad (46)$$

But, from (44), any constant $\mathbf{a} \equiv (\alpha_5, \alpha_6, \alpha_7) \neq 0$ implies $V_a = 0$. Thus, either $\mathbf{E} = 0$ or \mathbf{E} is orthogonal to \mathbf{a} in this particular case. However, since \mathbf{a} and \mathbf{E} are independent (*cf.* eqns (43) & (44)) then \mathbf{a} can be chosen to have arbitrary orientation relative to \mathbf{E} so that the possibility $\mathbf{E} \perp \mathbf{a}$ is excluded. Consequently, $V_a = 0$ implies $\mathbf{E} = 0$.

Second choice of basis: By contrast, if we fix the basis of (36) by choosing $(r, s) = (2, 3), (3, 1), (1, 2)$ and use $(F_{23}, F_{31}, F_{12}) \equiv (B_1, B_2, B_3)$ then, by similar arguments, $V_a = 0$ implies $\mathbf{B} = 0$ also.

To summarize: since $V_a = 0$ implies $\mathbf{E} = 0$ and $\mathbf{B} = 0$, then the absence of the massive vector field implies the absence of the electromagnetic field. Consequently, the massive vector field is *always* present in the presence of the electromagnetic field. The only possible conclusion is that the massive vector field must necessarily be identified with a classical non-zero mass photon.

9.3. Constraints for the massive photon

The object

$$V_a = \sum_{k=5}^7 P_{rs}^a \alpha_k(\mathbf{x}, ct) \quad (47)$$

has been identified as a non-zero mass photon from which it is clear that it has only three degrees of freedom expressed in terms of the three functions $\mathbf{a} \equiv (\alpha_5, \alpha_6, \alpha_7)$. In the following, we show that each of the components of \mathbf{a} must satisfy the Klein-Gordon equation.

With the basis $(r, s) = (1, 4), (2, 4), (3, 4)$, then (21) gives

$$V_a = P_{14}^a \alpha_{14} + P_{24}^a \alpha_{15} + P_{34}^a \alpha_{16}$$

which, after expanding the operators P_{rs}^a gives

$$(V_1, V_2, V_3, V_4) = [-X_4(\alpha_{14}, \alpha_{15}, \alpha_{16}), \nabla \cdot (\alpha_{14}, \alpha_{15}, \alpha_{16})]$$

↓

$$(\mathbf{V}, V_4) = [-X_4 \mathbf{a}, \nabla \cdot \mathbf{a}]. \quad (48)$$

Consequently, we find

$$\square^2(\mathbf{V}, V_4) = (-X_4^2 \mathbf{a}, \nabla \cdot \mathbf{a}). \quad (49)$$

We now consider how \mathbf{a} must be constrained to ensure (49) assumes the form of (42). A consideration of (11) shows that the most simple possibility is given by the condition $\square^2 \mathbf{a} = m \mathbf{a}$ for some parameter m since then, use of (48) reduces (49) to

$$\square^2(\mathbf{V}, V_4) = m(\mathbf{V}, V_4)$$

which is (4) without the conserved current. That is, the three functions, $\alpha_{14}, \alpha_{15}, \alpha_{16}$, which define the massive photon field V_a must each satisfy the Klein-Gordon equation.

10. Massive Photons and Mechanical Reaction

The eigenvalue $\lambda = 1$ is associated with two distinct three-dimensional subspaces of eigenvectors, $R_{sk,3}$ and $R_{sv,3}$ of which the first has been identified with the electromagnetic field and the second with a field of classical massive mass photons. The general solutions associated with each of these subspaces are given by

$$F_{ab} = \sum_{k=2}^4 U_{ab}^k \alpha_k(\mathbf{x}, ct),$$

$$G_{ab} = \sum_{k=5}^7 U_{ab}^k \alpha_k(\mathbf{x}, ct),$$

respectively. However, since they are *both* associated with $\lambda = 1$, then the *most* general solution associated with this particular eigenvalue is

$$\Psi_{ab} = F_{ab} + G_{ab}.$$

Now, according to the Lorentz force-law, the four-force generated by an electromagnetic field on a charged particle, e , with four-velocity \mathbf{V} is given by $F_a = eV_i F_{ai}/c$. Thus, if the idea of the Lorentz force is to be generalized, then the most general solution associated with $\lambda = 1$, $\Psi_{ab} \equiv F_{ab} + G_{ab}$ must give rise to a total system force of

$$F_a = \frac{e}{c} V_i F_{ai} + \frac{e}{c} V_i G_{ai}.$$

The natural question now is *what does $eV_i G_{ai}/c$ represent?* It is well known that, according to the Lorentz force law of classical electrodynamics, the net electromagnetic forces generated by two charged particles on each other are not equal and opposite—that is, even in the case of non-relativistic motions, the classical electrodynamic description of a mutually interacting charged particle-pair does not satisfy Newtonian conservation principles.

Consequently, *dynamical reactions*, and the freedom to include them, are missing from classical electrodynamics. Since we have already identified G_{ab} with an irreducible field of massive photons and since we know that an accelerated charged particle radiates electromagnetically, then the obvious interpretation is that the massive photons of the theory are these radiated photons and that $eV_i G_{ai}/c$ represents the reaction force associated with these accelerated photons; that is, it describes the *reaction* on the particle of charge e of its own action on the source of the field F_{ab} .

Thus, suppose that a non-relativistic system consists of just two charged particles, e_1 and e_2 with respective four-velocities $V_a^{(1)}$ and $V_a^{(2)}$, and that each particle generates electromagnetic fields $F_{ab}^{(1)}$ and $F_{ab}^{(2)}$ respectively, and generates a reaction to the action on itself through the reaction fields $G_{ab}^{(2)}$ and $G_{ab}^{(1)}$ respectively. Then, the respective forces acting in the vicinity of each particle are:

$$F_a^{(1)} = \frac{e_1}{c} V_i^{(1)} F_{ai}^{(2)} + \frac{e_1}{c} V_i^{(1)} G_{ai}^{(2)},$$

$$F_a^{(2)} = \frac{e_2}{c} V_i^{(2)} F_{ai}^{(1)} + \frac{e_2}{c} V_i^{(2)} G_{ai}^{(1)}.$$

If action and reaction are to be equal and opposite in this non-relativistic system, then we must have

$$F_a^{(1)} + F_a^{(2)} = 0$$

which, given the fields $F_{ab}^{(1)}$ and $F_{ab}^{(2)}$, represent a constraint on the reaction fields $G_{ab}^{(1)}$ and $G_{ab}^{(2)}$.

11. Conclusions

11.1. General comments

The work of this paper began by noting that classical electrodynamics possesses two generic properties—*Property A* and

Property B—and then searched for general theories possessing these properties simultaneously (*cf.* §1.2, 1.3). The analysis recovered the Maxwell field, as expected and required, but also gave the surprising result that it cannot exist in isolation, but must always be associated with an additional massive vector field. The irreducible nature of this association led us to identify this massive vector field as the classical representation of the massive photon.

This approach opens the door on many possibilities, and we have briefly discussed one of them: specifically, that the difficulties associated with the fact that the Lorentz force law does not conform to Newtonian ideals are removed when this law is generalized to account for the presence of the new vector field.

11.2. Relationship with the canonical viewpoint

Classical electrodynamics has arisen, primarily, as the synthesis of laboratory-based experience and, in its covariant four-vector formulation, the electromagnetic field has a very beautiful interpretation as the $U(1)$ -gauge field of the superbly successful quantum electrodynamics (qed) with the corresponding gauge particle being the *massless* photon. Thus, against the positive aspects of the ideas discussed herein, we must weigh the fact that the idea of the massive photon is radically at variance with classical qed. So, the question arises of whether it is possible to reconcile the results of this paper with qed and all that that theory represents. This author believes the answer to be positive, and argues as follows:

It was shown, in §6, that the new covariant formalism, based on the three-dimensional linear space of operators $(R)_{sk,3}$, is expressed purely in terms of the classical magnetic vector potential and makes no reference to the scalar potential. Thus, there can be no discussion of electrostatics in the $(R)_{sk,3}$ formalism. But the canonical theory, and hence the possibility _{$sk,3$} of electrostatics, is recovered by the introduction of an arbitrary scalar field—identifiable as the classical scalar potential—into the $(R)_{sk,3}$

formalism. We now note that the theory of electrostatics assumes the existence of a charge distribution, Q say, which is at rest in some particular inertial frame. The theory then considers the effects of Q on some other charge, q say, under the assumption that q does *not* affect the state of motion of Q . Thus, in effect, the theory of electrostatics is a test-particle theory—it is an idealization that, in literal practice, can be approached but never attained. We can then conclude that any theory for which the scalar potential is an essential component (e.g., the canonical covariant four-vector formalism) contains, at some fundamental level, the assumptions of a test-particle theory. Thus, we would argue that the covariant $(R)_{sk,3}$ formalism can be reconciled with the canonical four-vector formalism under the hypothesis that the transition from the former to the latter is the transition from a “real world” electromagnetism to its test-particle idealization.

11.3. Empirical evidence for non-zero mass photons?

The idea that photons might be massive is not new—Vigier [15], for example, argues that the long sequence of interferometer measurements made over several decades by Michelson, Morley and Miller [11,13,12,14] are actually consistent with a non-zero photon mass, and put an upper bound of about 10^{-68} kg on this mass. Vigier’s interest in this is well known since, as a one-time student of DeBroglie, he has long recognized that the latter’s interpretation of quantum mechanics probably requires photons to be massive—and *vice versa*, for if photons are found to have non-zero mass, then the standard interpretation of quantum mechanics—and with it, the whole standard model of particle physics—will have question marks raised over it.

11.4. Astrophysics

From an astrophysical point of view, the stakes are also high—the standard interpretation of cosmological redshift requires that it arise purely from expansion. But the only direct evidence supporting this interpretation comes from using the standard

candles to verify Hubble's law—but this can only be done out to very modest distances—this direct evidence is actually extrapolated over many decades when it comes to discussing the physics of very high redshift objects. The standard argument against non-expansion mechanisms is that there is no conceivable alternative mechanism which would not broaden spectra, nor leave images unblurred. Since spectral lines are sharp and since images are remarkably un-blurred, it is inferred that redshift must be an expansion effect. However, such arguments are predicated directly upon the notion of the massless photon. Once massive photons are admitted, many different alternative mechanism become, at least in principle, possible.

Appendix A. The Lagrangian density

The required Lagrangian density was arrived at via the following considerations:

- The identities which occur in the canonical covariant formulation of electromagnetism arise because, when F^{ab} is defined as it is from the four-vector potential, the field equation and the Jacobi identity define self-cancelling sums of permutations of fixed-order differential operations on the arbitrarily defined four-vector potential. It is straightforward to see that, in the general case, a necessary condition for a differential expression to become an identity in this way is that the expression concerned must be homogeneous in the differential operators it contains (*differentially homogeneous*) and, when such expressions arise from variational principles, then the corresponding Lagrangian densities must also be differentially homogeneous.

Since the necessary skew-symmetry of the electromagnetic field tensor in *any* covariant formulation ensures that the identity

$$\frac{\partial F_{ij}}{\partial x^i \partial x^j} = 0$$

is satisfied, then we are led to consider only those variational principles which give rise to equations which are second order in the field over which they are defined.

- Finally, in order to guarantee the algebraic orthogonality properties that we require, we must add in the general constraint that any variational principle must be invariant with respect to the interchange of any of its indices—this is also necessary to ensure that changing the labels of axes has no effect.

Putting these considerations together, the most general density is given by

$$\begin{aligned} L = & \alpha_0 \frac{\partial \Psi_{ij}}{\partial x^k} \frac{\partial \Psi_{ji}}{\partial x^k} + \alpha_1 \frac{\partial \Psi_{ij}}{\partial x^k} \frac{\partial \Psi_{ij}}{\partial x^k} \\ & + \beta_0 \frac{\partial \Psi_{ik}}{\partial x^i} \frac{\partial \Psi_{kj}}{\partial x^j} + \beta_1 \left(\frac{\partial \Psi_{ik}}{\partial x^i} \frac{\partial \Psi_{jk}}{\partial x^j} + \frac{\partial \Psi_{ki}}{\partial x^i} \frac{\partial \Psi_{kj}}{\partial x^j} \right) \\ & + \gamma_0 \frac{\partial \Psi_{ik}}{\partial x^j} \frac{\partial \Psi_{kj}}{\partial x^i} + \gamma_1 \left(\frac{\partial \Psi_{ik}}{\partial x^j} \frac{\partial \Psi_{jk}}{\partial x^i} + \frac{\partial \Psi_{ki}}{\partial x^j} \frac{\partial \Psi_{kj}}{\partial x^i} \right), \end{aligned}$$

where $(\alpha_0, \alpha_1, \beta_0, \beta_1, \gamma_0, \gamma_1)$ are arbitrary constants. This is easily shown to contain a large amount of redundancy.

Appendix B. The Orthogonality Relations

The orthogonality relations within the system are given because they provide an insight into the nature of Maxwell's equations, as we have seen in §9.

The algebraic eigensystem (11) is given by

$$\left(\sigma^{ij} X_i X_j \right) \mathbf{U}_k = \lambda \mathbf{U}_k$$

where $\mathbf{U}^k = (U_{11}^k, U_{12}^k, U_{13}^k, \dots, U_{44}^k)$, where σ^{ab} ($a, b = 1 \dots 4$) are sixteen matrices each of dimension 16×16 whose elements in row (i, j) and column (r, s) are given by

$$\sigma_{ij:rs}^{ab} = \delta_{ia} \delta_{rb} \delta_{sj} + \delta_{ja} \delta_{sb} \delta_{ri}, \quad (\text{B.1})$$

where the columns are taken in order $(1,1), (1,2), (1,3), (1,4), (2,1) \dots (4,4)$ and where δ_{ab} is the 4×4 unit matrix. These matrices are easily shown to be symmetric so that, consequently, we have

$$\left(\sigma^{ij} X_i X_j \right)^T = \left(\sigma^{ij} X_i X_j \right).$$

We can conclude from this that eigenvectors lying in distinct subspaces of the eigenspace are orthogonal with respect to the ordinary vector scalar product; that is, if $\mathbf{U}_A, \mathbf{U}_B, \mathbf{U}_C, \mathbf{U}_D$ and \mathbf{U}_E are such that

$$\mathbf{U}_A \in (G)_{sy,6}; \quad \mathbf{U}_B \in (G)_{sk,3};$$

$$\mathbf{U}_C \in (R)_{sy,3}; \quad \mathbf{U}_D \in (R)_{sk,3}; \quad \mathbf{U}_E \in (R)_{sy,1},$$

then

$$\mathbf{U}_A^T \mathbf{U}_B = \mathbf{U}_A^T \mathbf{U}_C = \mathbf{U}_A^T \mathbf{U}_D = \mathbf{U}_A^T \mathbf{U}_E = 0$$

$$\mathbf{U}_B^T \mathbf{U}_C = \mathbf{U}_B^T \mathbf{U}_D = \mathbf{U}_B^T \mathbf{U}_E = 0$$

$$\mathbf{U}_C^T \mathbf{U}_D = \mathbf{U}_C^T \mathbf{U}_E = 0 \quad (\text{B.2})$$

$$\mathbf{U}_D^T \mathbf{U}_E = 0,$$

where, by $\mathbf{U}^T \mathbf{U}$, we effectively mean $U_{ij} U_{ij}$, with summation over the indices (i, j).

Naturally, these relations are directly verifiable by direct reference to the definitions of the eigenvectors given at (12), (16), (15), (13) and (14) respectively.

Appendix C. A Material Vacuum?

It has been shown how, according to Poincaré-invariant electrodynamics, the magnetic vector potential supports a stationary longitudinal wave, as well as the conventional propagating transverse wave. The classical theory has managed to assimilate the idea of an electrodynamic wave propagating in the absence of any supporting medium—if only because there is, at least, a sense of *something* travelling and things can be conceived as travelling through empty space. However, the idea of stationary longitudinal waves in the absence of any kind of supporting medium presents an entirely new level of incomprehensibility since, now, *nothing* is travelling anywhere. In Appendix D, a possible solution to this perceived problem is obtained by showing how (29) supports a non-wavy $\mathbf{J} = 0$ solution which has a ready interpretation as a classical description of a fluctuating material vacuum. The magnetic vector potential waves discussed in §8 can then interpreted as *disturbances* of this material vacuum.

Appendix D. The Material Vacuum

For $\mathbf{J} = 0$ and defining $\mathbf{x} \equiv (x_1, x_2, x_3)$, it is easily shown how (29) has a relativistically invariant non-radiated solution, given by

$$\mathbf{A} = (\Delta_0 - \Delta_1) \mathbf{A}_{01},$$

$$\Delta_0 \equiv (\mathbf{x} - \mathbf{x}_0)^2 - c^2(t - t_0)^2,$$

$$\Delta_1 \equiv (\mathbf{x} - \mathbf{x}_1)^2 - c^2(t - t_1)^2,$$

for an arbitrary constant vector \mathbf{A}_{01} and origins (\mathbf{x}_0, ct_0) and (\mathbf{x}_1, ct_1) which satisfy $\Delta_0 > 0$ and $\Delta_1 > 0$ but which are otherwise arbitrary. Consequently, the general solution of this type is given by

$$\mathbf{A}_{vac} = \sum_{\mathbf{x}_0, ct_0} \sum_{\mathbf{x}_1, ct_1} (\Delta_0 - \Delta_1) \mathbf{A}_{01}(\mathbf{x}_0, ct_0, \mathbf{x}_1, ct_1) \quad (\text{D.1})$$

where the summation is intended to be over all admissible spacetime origins, (\mathbf{x}_0, ct_0) and (\mathbf{x}_1, ct_1) and it has been assumed that the constant vector \mathbf{A}_{01} , which is a function of these origins, is such that the summation is uniformly convergent.

An understanding of the meaning of this solution can be had by considering the expanding surface

$$(\mathbf{x} - \mathbf{x}_0)^2 - c^2(t - t_0)^2 = k^2,$$

for k some real constant, generated by a single term in (D1). It is easily shown that the radial speed of such an expanding surface increases from 0 to c on the range $|k| \leq |\mathbf{x} - \mathbf{x}_0| < \infty$. It follows that \mathbf{A}_{vac} , which is defined by (D.1) at the spacetime point (\mathbf{x}, ct) by summing over all admissible origins (\mathbf{x}_0, ct_0) and (\mathbf{x}_1, ct_1) , is a sum over an infinity of instantaneously intersecting surfaces expanding from all possible directions and at all possible subluminal speeds. In this way, we generate a classical image of a continually fluctuating relativistically invariant material vacuum; for example, see Dirac [10]. Consequently, it is suggested that the solution (D.1) can be interpreted as a classical model of the material vacuum within which magnetic vector potential waves can be interpreted as disturbances. It is to be noted that this material vacuum model is also applicable in the conventional electrodynamic theory.

Finally, it should be remarked that the non-radiated field \mathbf{A}_{vac} should give rise to non-zero electromagnetic effects through (19). The reality, which is that such effects are only found at extremely low levels in the quantum fluctuations of the vacuum, puts constraints on the constants in (D.1).

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