Quaternions, Maxwell Equations and Lorentz Transformations

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In this work: a).-We show that the invariance of the Maxwell equations under duality rotations brings into scene to the complex vector $(c\vec{B}+i\vec{E})$, whose components allow to construct a quaternionic equation for the electromagnetic field in vacuo. b).-For any analytic function f of the complex variable z, it is possible to prove that is a Debye potential for itself, which permits to reformulate the corresponding Cauchy-Riemann relations. Here we show that the Fueter conditions- when z is a quaternion- also accept a similar reformulation and a very compact quaternionic expression. c).-We exhibit how the rotations in three and four dimensions can be described through a complex matrix relation or equivalently by a quaternionic formula.

1. Quaternionic version of the Maxwell equations.

The Maxwell equations in the source-free case:

$$\vec{\nabla} \bullet \vec{B} = 0, \qquad \vec{\nabla} \bullet \vec{E} = 0,$$
$$\vec{\nabla} \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{\mathrm{E}}}{dt}, \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{1}$$

are invariant under the duality rotations [1,2]:

$$\vec{\mathrm{E}}' = \vec{\mathrm{E}}Cos\alpha + c\vec{B}Sin\alpha, \qquad \mathrm{c}\vec{B}' = -\vec{E}Sin\alpha + c\vec{B}Cos\alpha, \tag{2}$$

in the sense that the fields also satisfy (1) ; the Noether theorem [3-9] shows [10] that this invariance of the Maxwell equations implies the continuity equation:

$$\frac{\partial}{\partial t}\left(\frac{\in_0}{2}E^2 + \frac{1}{2\mu_0}B^2\right) + \vec{\nabla} \bullet \left(\frac{1}{\mu_0}\vec{E} \times \vec{B}\right) = 0, \tag{3}$$

for the electromagnetic energy.If relations (2) are rewritten into the form:

$$c\vec{B}' + i\vec{E}' = e^{i\alpha}\left(c\vec{B} + i\vec{E}\right), \tag{4}$$

the participation of the complex vector [10-13]:

$$\vec{F} = c\vec{B} + i\vec{E} \tag{5}$$

follows, and expressions (1) become:

$$\vec{\nabla} \bullet \vec{F} = 0, \qquad \frac{1}{c}\frac{\partial \vec{F}}{\partial t} - i\vec{\nabla} \times \vec{F} = 0 \tag{6}$$

Now we show that the Maxwell equations adopt a very compact structure if we employ quaternions [10,14-23]. In fact, with we construct the quaternionic vector :

$$\boldsymbol{F} = \boldsymbol{I}F_X + \boldsymbol{J}F_Y + \boldsymbol{K}F_Z, \tag{7}$$

and the quaternonic operator [24-26]:

$$\nabla = \frac{i}{c}\frac{\partial}{\partial t} + \boldsymbol{I}\frac{\partial}{\partial x} + \boldsymbol{J}\frac{\partial}{\partial y} + \boldsymbol{K}\frac{\partial}{\partial \boldsymbol{z}}, \tag{8}$$

so that the Maxwell equations (1) are carried to the following quaternionic version:

$$\nabla \boldsymbol{F} = \boldsymbol{0}, \tag{9}$$

Conway [27] – Silberstein [28] introduced quaternions as a notation in the special theory of relativity; Silberstein [24]-Lanczos [25,29] were the first authors to deduce (9) (this quaternionic expression reminds us of the Weyl equation of massless ½ spin particles).

Unitary complex quaternions generate [10, 22, 30-33] proper Lorentz transformations, consequently, we consider as a natural fact to use quaternions – as in eq.(9) – for the description of the Maxwell field.

2. The Fueter conditions as Debye expressions

If *f* is an analytic function of the complex variable $z=x+iy$, then it has the form $f(z)=u(x,y)+iv(x,y)$ with the fulfillment of the Cauchy-Riemann relations [34]:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \tag{10}$$

which thereby imply the harmonic character of u and v because:

$$\nabla^2 u = \nabla^2 v = 0,\ \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{11}$$

The conditions (10) allow to obtain two interesting differential identities for *u* and *v*, which have great similarity with the Debye expressions [2, 35-39] for the electromagnetic potentials, in fact:

$$u = \frac{\vec{r}}{r} \bullet \vec{\nabla}(ru) - \left[\vec{r} \times \vec{\nabla} u\right]_3, \tag{12}$$

where we have employed the known notation from vectorial analysis:

$$\vec{r} = x\hat{i} + y\hat{j}, \quad r = \sqrt{x^2 + y^2},$$

$$\left[\vec{r} \times \vec{\nabla} g\right]_3 \equiv x\frac{\partial g}{\partial y} - y\frac{\partial g}{\partial x}, \quad \vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}, \tag{13}$$

The function if is also analytic, then *if(z)=-v+iu* implies that (12) is correct with the changes $u \to -v$ and $v \to u$, that is:

$$v = \frac{\vec{r}}{r} \bullet \vec{\nabla}(rv) + \left[\vec{r} \times \vec{\nabla} u\right]_3, \tag{14}$$

The expressions (12) and (14) are a reformulation of the Cauchy-Riemann relations, these being a strong motivation for the existence of Debye generators in electromagnetic theory. The solution of the source-free Maxwell equations can be written [2,35-39] in terms of two real scalar generators (Debye potentials) - ψ_E and ψ_M - which satisfy the wave equation:

$$\Box\psi_E = \Box\psi_M = 0, \qquad \Box = \frac{\partial^2}{c^2\partial t^2} - \nabla^2 \tag{15}$$

in according to:

$$\phi = -c\frac{\vec{r}}{r} \bullet \vec{\nabla}\left(r\psi_E\right), \qquad \vec{A} = -\vec{r} \times \vec{\nabla}\psi_M + \vec{r}\ \frac{\partial \psi_E}{c\partial t}, \tag{16}$$

up to gauge transformations. We must note that the existence of and implicitly follows from results of several authors [40-43].

Now we shall obtain the generalization of (12) and (14) for the quaternionic case. Fueter[44] founded the theory of functions $\boldsymbol{G}(\boldsymbol{q}) = \mathrm{u}_o + \boldsymbol{I}\mathrm{u}_1 + \boldsymbol{J}\mathrm{u}_2 + \boldsymbol{K}\mathrm{u}_3$, of a quaternionic variable $\boldsymbol{q} = \boldsymbol{x}_0 + \boldsymbol{I}\mathrm{y}_1 + \boldsymbol{J}\mathrm{y}_2 + \boldsymbol{K}\mathrm{y}_3$, and he imposed the following differential conditions on the , which correspond to the extension of the Cauchy-Riemann equations (10):

$$
\begin{aligned}
&\frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial y_1} - \frac{\partial u_2}{\partial y_2} - \frac{\partial u_3}{\partial y_3} = 0 \quad , \\
&\frac{\partial u_1}{\partial x_0} + \frac{\partial u_0}{\partial y_1} + \frac{\partial u_3}{\partial y_2} - \frac{\partial u_2}{\partial y_3} = 0 \quad , \\
&\frac{\partial u_2}{\partial x_0} - \frac{\partial u_3}{\partial y_1} + \frac{\partial u_0}{\partial y_2} + \frac{\partial u_1}{\partial y_3} = 0 \quad , \\
&\frac{\partial u_3}{\partial x_0} + \frac{\partial u_2}{\partial y_1} - \frac{\partial u_1}{\partial y_2} + \frac{\partial u_0}{\partial y_3} = 0 \quad .
\end{aligned}
\tag{17}
$$

Imaeda [45] shows that (17) permits to establish a connection with the Maxwell equations, which leads to a new formulation of classical electrodynamics. If we introduce the operator (8):

$$
\nabla = \frac{\partial}{\partial x_0} + \boldsymbol{I}\frac{\partial}{\partial y_1} + \boldsymbol{J}\frac{\partial}{\partial y_2} + \boldsymbol{K}\frac{\partial}{\partial y_3} \tag{18}
$$

then (17) are equivalent to:

$$
\nabla \boldsymbol{G} = \boldsymbol{0} \, , \tag{19}
$$

It is remarkable the similarity between (9) and (19), of course we may see to (9) as a particular case of (19).

With the aid of (17) and taking as guide the relation (12), it is not difficult to deduce the Debye type expression:

$$u_0 = \frac{\vec{r}}{r} \bullet \vec{\nabla}(r u_0) + \frac{\partial}{\partial x_0}(y_1 u_1 + y_2 u_2 + y_3 u_3) + \left[\vec{r} \times \vec{\nabla} u_1\right]_1 + \left[\vec{r} \times \vec{\nabla} u_2\right]_2 + \left[\vec{r} \times \vec{\nabla} u_3\right]_3 , \quad (20)$$

where

$$\vec{r} = \hat{i} y_1 + \hat{j} y_2 + \hat{k} y_3 , \quad \vec{\nabla} = \hat{i}\frac{\partial}{\partial y_1} + \hat{j}\frac{\partial}{\partial y_2} + \hat{k}\frac{\partial}{\partial y_3} , \quad \left[\vec{r} \times \vec{\nabla} g\right]_1 \equiv y_2 \frac{\partial g}{\partial y_3} - y_3 \frac{\partial g}{\partial y_2} ,$$
$$\left[\vec{r} \times \vec{\nabla} g\right]_2 \equiv y_3 \frac{\partial g}{\partial y_1} - y_1 \frac{\partial g}{\partial y_3} , \quad \left[\vec{r} \times \vec{\nabla} g\right]_3 \equiv y_1 \frac{\partial g}{\partial y_2} - y_2 \frac{\partial g}{\partial y_1} . \quad (21)$$

The function $-\boldsymbol{G}(\boldsymbol{q})\boldsymbol{I} = u_1 - \boldsymbol{I}u_0 - \boldsymbol{J}u_3 + \boldsymbol{K}u_2$ - is also analytic, then in (20) we can make the changes $u_0 \to u_1$, $u_1 \to -u_0$, $u_2 \to -u_3$ and $u_3 \to u_2$, therefore:

$$u_1 = \frac{\vec{r}}{r} \bullet \vec{\nabla}(r u_1) + \frac{\partial}{\partial x_0}(-y_1 u_0 - y_2 u_3 + y_3 u_2) - \left[\vec{r} \times \vec{\nabla} u_0\right]_1 - \left[\vec{r} \times \vec{\nabla} u_3\right]_2 + \left[\vec{r} \times \vec{\nabla} u_2\right]_3 . \quad (22)$$

Similarly the analytic character of **-G**$(\mathbf{q})$**J** and **-G**$(\mathbf{q})$**K** leads to:

$$u_2 = \frac{\vec{r}}{r} \bullet \vec{\nabla}(r u_2) + \frac{\partial}{\partial x_0}(y_1 u_3 - y_2 u_0 - y_3 u_1) + \left[\vec{r} \times \vec{\nabla} u_3\right]_1 - \left[\vec{r} \times \vec{\nabla} u_0\right]_2 - \left[\vec{r} \times \vec{\nabla} u_1\right]_3 ,$$
$$u_3 = \frac{\vec{r}}{r} \bullet \vec{\nabla}(r u_3) + \frac{\partial}{\partial x_0}(-y_1 u_2 + y_2 u_1 - y_3 u_0) - \left[\vec{r} \times \vec{\nabla} u_2\right]_1 - \left[\vec{r} \times \vec{\nabla} u_1\right]_2 - \left[\vec{r} \times \vec{\nabla} u_0\right]_3 . \quad (23)$$

The relations (20), (22) and (23) represent a Debye type reformulation of the Fueter conditions (17), which are relations not explicitly found in the literature.

3. Quaternions, 3-rotations and Lorentz transformations

In Minkowski space, any real matrix $\underset{\sim}{L}_{4\times 4}=\left(L_{jk}\right)$ with the property $\underset{\sim}{L}^T\underset{\sim}{L}=\underset{\sim}{I}$, that is:

$$L_{jk}L_{jl}=\delta_{kl} \quad , \tag{24}$$

allows to make a Lorentz transformation over an arbitrary event , via the expression [22,41]:

$$x'_j=L_{jk}x_k \tag{25}$$

such that (24) implies the invariance $x'_j\,x'_j=x_jx_j$, this being:

$$x'^2+y'^2+z'^2-c^2t'^2=x^2+y^2+z^2-c^2t^2. \tag{26}$$

If we define the complex 2x2 matrices:.

$$\underset{\sim}{X}=\begin{pmatrix} x_3-ix_4 & x_1-ix_2 \\ x_1+ix_2 & -x_3-ix_4 \end{pmatrix},\underset{\sim}{U}=\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \tag{27}$$

with the properties:

$$\det\ \underset{\sim}{X}=-x_jx_j \quad , \qquad \underset{\sim}{\mathrm{U}}^\dagger=\begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix} \quad , \tag{28}$$

then the construction of a Lorentz transformation $\underset{\sim}{L}$ can be accomplished through the relation [22,41]:

$$\underset{\sim}{X}'=\underset{\sim}{U}\underset{\sim}{X}\underset{\sim}{\mathrm{U}}^\dagger \quad , \tag{29}$$

with $\det\underset{\sim}{U}=1$ as required by (26). In other words, any four complex constants $\alpha,\beta,\gamma,\delta$ subject to the unimodular condition:

$$\alpha\delta-\beta\gamma=1 \tag{30}$$

generate also a Lorentz's matrix. By comparison between (25) and (29) it results the following expressions [46] of Synge[41] – Rumer[47]- Aharoni[48]:

$$
\begin{aligned}
&L_{11}=\frac{1}{2}\left(\alpha^*\delta+\beta\gamma^*\right)+c.c. \quad , && L_{12}=\frac{i}{2}\left(\alpha^*\delta+\beta\gamma^*\right)+c.c. \quad ,\\
&L_{13}=\frac{1}{2}\left(\alpha^*\gamma-\beta^*\delta\right)+c.c. \quad , && L_{14}=\frac{1}{2}\left(\alpha^*\gamma+\beta^*\delta\right)+c.c. \quad ,\\
&L_{21}=\frac{i}{2}\left(\alpha\delta^*-\beta\gamma^*\right)+c.c. \quad , && L_{22}=\frac{1}{2}\left(\alpha^*\delta-\beta^*\gamma\right)+c.c. \quad ,\\
&L_{23}=\frac{i}{2}\left(\alpha\gamma^*+\beta^*\delta\right)+c.c. \quad , && L_{24}=\frac{i}{2}\left(\alpha\gamma^*-\beta^*\delta\right)+c.c. \quad ,\\
&L_{31}=\frac{1}{2}\left(\alpha^*\beta-\gamma^*\delta\right)+c.c. \quad , && L_{32}=\frac{i}{2}\left(\alpha^*\beta-\gamma^*\delta\right)+c.c. \quad ,\\
&L_{33}=\frac{1}{2}\left(\alpha\alpha^*-\beta\beta^*-\gamma\gamma^*+\delta\delta^*\right) \quad , && L_{34}=\frac{1}{2}\left(\alpha\alpha^*+\beta\beta^*-\gamma\gamma^*-\delta\delta^*\right) \quad ,\\
&L_{41}=\frac{1}{2}\left(\alpha^*\beta+\gamma^*\delta\right)+c.c. \quad , && L_{42}=\frac{i}{2}\left(\alpha^*\beta+\gamma^*\delta\right)+c.c. \quad ,\\
&L_{43}=\frac{1}{2}\left(\alpha\alpha^*-\beta\beta^*+\gamma\gamma^*-\delta\delta^*\right) \quad , && L_{44}=\frac{1}{2}\left(\alpha\alpha^*+\beta\beta^*+\gamma\gamma^*+\delta\delta^*\right) \quad ,
\end{aligned}
\tag{31}
$$

where c.c. means the complex conjugate of all the previous terms. It is evident that the matrices produce the same, thus they are said [32, 33, 49, 50] to constitute a two-valued representation of the Lorentz transformations.

On the other hand, we may follow Lanczos [10, 22] and introduce the quaternions [10,14-22]:

$$\mathbf{R}=ct+i(\mathbf{I}x+\mathbf{J}y+\mathbf{K}z) \quad , \qquad \mathbf{A}=a_4+\mathbf{I}a_1+\mathbf{J}a_2+\mathbf{K}a_3 \quad , \tag{32}$$

together with the definitions:

$$\bar{\mathbf{A}} = a_4 - \left(\mathbf{I}a_1 + \mathbf{J}a_2 + \mathbf{K}a_3\right) \quad , \qquad \mathbf{A}^* = a^*{}_4 + \mathbf{I}a^*{}_1 + \mathbf{J}a^*{}_2 + \mathbf{K}a^*{}_3 \quad , \tag{33}$$

so that A generates a Lorentz's matrix via the quaternionic relation:

$$\mathbf{R}' = \mathbf{A}\mathbf{R}\bar{\mathbf{A}}^* \tag{34}$$

with A fulfilling the condition:

$$\mathbf{A}\bar{\mathbf{A}} = a_1^2 + a_2^2 + a_3^2 + a_4^2 = 1 \ . \tag{35}$$

For example, (35) is verified by:

$$\begin{aligned} a_1 &= -\frac{i}{2}(\gamma + \beta) \quad , \qquad a_2 = -\frac{1}{2}(\gamma - \beta) \quad , \\ a_3 &= \frac{i}{2}(\delta - \alpha) \quad , \qquad a_4 = -\frac{1}{2}(\delta + \alpha) \quad , \end{aligned} \tag{36}$$

and, if the complex numbers satisfy (30) then (34) and (36) imply (31). Another option is:

$$\begin{aligned} a_1 &= iQ\left(\lambda^* e^{\underline{P}} + \eta^* e^{-\underline{P}}\right) \quad , \qquad a_2 = Q\left(\lambda^* e^{\underline{P}} - \eta^* e^{-\underline{P}}\right) \quad , \\ a_3 &= iQ\left(e^{\underline{P}} - e^{-\underline{P}}\right) \quad , \qquad a_4 = -Q\left(e^{\underline{P}} + e^{-\underline{P}}\right) \quad , \\ \underline{P} &= \frac{1}{2}(M + iN) \quad , \qquad Q = \frac{1}{2}\left|1 - \lambda^*\eta^*\right|^{-1/2} \quad , \end{aligned} \tag{37}$$

where M,N are arbitrary real numbers, and λ, η are any complex numbers such that $\lambda\eta \neq 1$. Eqs. (34) and (37) give us the following expressions [46] of Greenberg-Knauer [51] for $\underset{\sim}{L}$:

$$
\begin{aligned}
&L_{11} = Te^{iN}\left(1+\lambda^{*}\eta\right)+c.c. \quad , && L_{12} = iTe^{iN}\left(1-\lambda^{*}\eta\right)+c.c. \quad ,\\
&L_{13} = Te^{iN}\left(\eta-\lambda^{*}\right)+c.c. \quad , && L_{14} = -Te^{iN}\left(\eta+\lambda^{*}\right)+c.c. \quad ,\\
&L_{21} = iTe^{-iN}\left(1+\lambda\eta^{*}\right)+c.c. \quad , && L_{22} = -Te^{-iN}\left(\lambda\eta^{*}-1\right)+c.c. ,\\
&L_{23} = iTe^{-iN}\left(\eta^{*}-\lambda\right)+c.c. \quad , && L_{24} = -iTe^{-iN}\left(\eta^{*}+\lambda\right)+c.c. \quad ,\\
&L_{31} = T\left(\lambda e^{M}-\eta e^{-M}\right)+c.c. \; , && L_{32} = iT\left(\lambda e^{M}+\eta e^{-M}\right)+c.c. \; ,\\
&L_{33} = T\left[e^{M}\left(1-\lambda\lambda^{*}\right)+e^{-M}\left(1-\eta\eta^{*}\right)\right] \; , && L_{34} = -T\left[e^{M}\left(1+\lambda\lambda^{*}\right)-e^{-M}\left(1+\eta\eta^{*}\right)\right] \; ,\\
&L_{41} = -T\left(\lambda e^{M}+\eta e^{-M}\right)+c.c. \quad , && L_{42} = -iT\left(\lambda e^{M}+\eta e^{-M}\right)+c.c. \quad ,\\
&L_{43} = -T\left[e^{M}\left(1-\lambda\lambda^{*}\right)-e^{-M}\left(1-\eta\eta^{*}\right)\right] \; , && L_{44} = T\left[e^{M}\left(1+\lambda\lambda^{*}\right)+e^{-M}\left(1+\eta\eta^{*}\right)\right] \; ,
\end{aligned} \tag{38}
$$

with $T=\frac{1}{2}\left|1-\lambda\eta\right|^{-1}$ well behaved because $\lambda\eta \neq 1$. Sachs [52] obtained some special cases of (38). The refs. [50, 53] have important applications of (38) to the Newman-Penrose formalism [54, 55] in general relativity.

Now we consider that A is a real unitary quaternion with their four components a_j written in terms of two complex numbers α and β :

$$
\begin{aligned}
&a_1 = -\frac{i}{2}\left(\beta-\beta^{*}\right) \quad , \qquad a_2 = -\frac{1}{2}\left(\beta+\beta^{*}\right) \quad ,\\
&a_3 = \frac{i}{2}\left(\alpha-\alpha^{*}\right) \quad , \qquad a_4 = -\frac{1}{2}\left(\alpha+\alpha^{*}\right) \quad ,\\
&\alpha\alpha^{*}+\beta\beta^{*}=1
\end{aligned} \tag{39}
$$

then (34) implies a rotation in the 3-space:

$$
\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \equiv \underset{\sim}{R} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad , \quad t' = t \quad , \tag{40}
$$

where [47,56]:

$$\underset{\sim}{R}=\begin{pmatrix} \frac{1}{2}\left(\alpha^2+\alpha^{*2}-\beta^2-\beta^{*2}\right) & -\frac{i}{2}\left(\alpha^2-\alpha^{*2}+\beta^2-\beta^{*2}\right) & -\left(\alpha\beta+\alpha^*\beta^*\right) \\ \frac{1}{2}\left(\alpha^2-\alpha^{*2}-\beta^2+\beta^{*2}\right) & \frac{1}{2}\left(\alpha^2+\alpha^{*2}+\beta^2+\beta^{*2}\right) & -i\left(\alpha\beta-\alpha^*\beta^*\right) \\ \alpha\beta^*+\alpha^*\beta & i\left(\alpha\beta^*-\alpha\beta^*\right) & \alpha\alpha^*-\beta\beta^* \end{pmatrix} \quad (41)$$

is an orthogonal matrix (element of O(3)) because:

$$\underset{\sim}{R}\underset{\sim}{R}^T=\underset{\sim}{I} \quad . \qquad (42)$$

The representation of an arbitrary rotation of three-space with the help of a real quaternion of length 1 was known by Euler and it was employed by him [10, 57].

It is interesting to note that if we make $x_4=0, \gamma=-\beta^*, \delta=\alpha^*$ into (29), then we obtain (40) and (41), being $\underset{\sim}{U}$ an element of SU (2) because:

$$\underset{\sim}{U}=\begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \quad , \quad \underset{\sim}{U}\underset{\sim}{U}^\dagger=\underset{\sim}{I} \quad , \qquad (43)$$

$$\det\underset{\sim}{U}=\alpha\alpha^*+\beta\beta^*=1 \quad .$$

The unitary matrices $\pm\underset{\sim}{U}$ generate the same orthogonal matrix $\underset{\sim}{R}$, thus SU (2) is a two-valued representation of O(3) [32, 33, 56, 58-62]. On the other hand, (39) is equivalent to:

$$\alpha=a_4-ia_3 \quad , \quad \beta=-a_2-ia_1 \quad . \qquad (44)$$

so that takes the form:

$$\underset{\sim}{U} = a_4 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - ia_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - ia_2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - ia_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad (45)$$
$$= a_4 \underset{\sim}{I} - i\left(a_1\sigma_x + a_2\sigma_y + a_3\sigma_z\right),$$

where $\sigma_x, \sigma_y, \sigma_z$ are the known matrices of Pauli. If now we use the formal association [21]:

$$\underset{\sim}{I} \to 1 \quad , \quad -i\sigma_x \to \mathbf{I} \quad , \quad -i\sigma_y \to \mathbf{J} \quad , \quad -i\sigma_z \to \mathbf{K} \qquad (46)$$

it follows that $\underset{\sim}{U} \to \mathbf{A}$, which motivates the intimate relationship between complex 2x2 unitary matrices and real 3x3 orthogonal matrices generated by real quaternions of length 1.

Thus we have seen that (29) or (34) describe completely the rotations in three and four dimensions.

References

[1] J. A. Wheeler, Geometrodynamics, Academic Press, NY (1962)

[2] G.F.Torres del Castillo, Rev. Mex. Fis. 43 (1997) 25

[3] E. Noether, Nachr. Ges. Wiss. Göttingen (1918) 235

[4] H. Weyl, Scripta Math. III 3 (1935) 201

[5] A. Trautman, Commun. Math. Phys. 6 (1967) 248

[6] C. Lanczos, Mathematical methods in solid state Physics and superfluid theory, Eds.R.C. Clark and G. H. Derik, Oliver & Boyd: Edinburgh (1969)

[7] C. H. Kimberling, Am. Math. Mon. 79 (1972) 136

[8] C. Lanczos, Bull. Inst. Math. and Appl. 9 (1973) 253

[9] N. Byers, Emmy Noether´s discovery of the deep connection between symmetries and conservation laws, Proc. Symp. Heritage E. Noether, Bar-Ilan Univ., Israel, Dec. (1996)

[10] C.Lanczos, The variational principles of mechanics, University of Toronto Press (1970)

[11] A. Sudbery, J. Phys. A19, No.2 (1986) L33

[12] V.V. Dvoeglazov, Hadronic J. 16 (1993) 459

[13] H. Munera and O. Guzman, Mod. Phys. Lett.A12 (1997) 2089
[14] W.R. Hamilton, Phil. Mag. 25 (1844) 489
[15] W.R. Hamilton, Lectures on quaternions, Hudges & Smith, Dublin (1853)
[16] P. G. Tait, An elementary treatise on quaternions, Oxford Univ. Press (1875)
[17] W. K. Clifford, Proc. London Math. Soc. 4 (1876) 381
[18] P.Weiss, Proc. R.Irish Acad. 46 (1941) 129
[19] R.J. Stephenson, Am.J. Phys. 34 (1966) 194
[20] A.M. Bork, Am. J. Phys. 34 (1966) 202
[21] J. Kronsbein, Am. J. Phys. 35(1967) 335
[22] J.L. Synge, Comm. Dublin Inst. Adv. Stud. Ser. A, No.21 (1972)
[23] S.L. Altmann, Math. Mag. 62 (1989) 291
[24] L. Silberstein, Theory of relativity, Macmillan, London (1924)
[25] C. Lanczos, Zeits. Phys. 57 (1929) 447
[26] C.R. Moisil, Bull. Sci. Math. 55 (1931) 168
[27] A. W. Conway, Proc. R. Irish Acad. A29 (1911) 1
[28] L. Silberstein, Phil. Mag. 23 (1912) 790
[29] G. Marx, Lecture at the C. Lanczos Intal. Centenary Conf., Raleigh N.C. USA, Dec.(1993)
[30] P.A.M. Dirac, Proc. R. Irish Acad. A50 (1945) 261
[31] A.W. Conway, Proc. R. Soc. London A191 (1947) 137
[32] S.L. Altmann, Rotations, quaternions and double groups, Clarendon Press, Oxford (1986)
[33] K.N. Srinivasa Rao, The rotation and Lorentz groups and their representations for physicists, Wiley Eastern Limited-India (1988)
[34] R.V. Churchill, Complex Variable and applications, McGraw-Hill, NY(1960)
[35] W.B. Campbell and T. Morgan, Physica 53 (1971) 264
[36] C. G. Gray, Am. J. Phys. 46 (1978) 169
[37] A. C. T. Wu, Phys. Rev. D34 (1986) 3109
[38] G.F.Torres del Castillo, Rev. Mex. Fis. 35 (1989) 282
[39] G.F. Torres del Castillo, Rev. Mex. Fis. 38 (1992) 753
[40] R.Penrose, Proc. R. Soc. London A284 (1965) 159
[41] J.L.Synge, Relativity: the special theory, North-Holland, Amsterdan (1965)

[42] P.A.Hogan, J. Math. Phys. 28 (1987) 2089

[43] J.López-Bonilla, J.Morales, G.Ovando and J.J.Peña, Indian J.Phys. B74 (2000) 393

[44] R. Fueter, Comm. Math. Helv. 7 (1934-35) 307, 8 (1936-37) 371, 9 (1936-37)320 and 10 (1937-38) 327

[45] K. Imaeda, Nuovo Cim. B32 (1976) 138

[46] J. López Bonilla, J. Morales and G. Ovando, Bull. Allahabad Math. Soc. 17 (2002) 53

[47] Ju. Rumer, Spinorial analysis, Moscow (1936)

[48] J. Aharoni, The special theory of relativity, Clarendon Press, Oxford (1959)

[49] W. Rindler, Am. J. Phys. 35 (1967) 937

[50] M. Carmeli, Group theory and general relativity, McGraw-Hill, NY (1977)

[51] P.J. Greenberg and J.P. Knauer, Stud. Appl. Math. 53 (1974) 165

[52] R.K. Sachs, Proc. R. Soc. London A264 (1961) 309

[53] D. Kramer, H. Stephani, M. MacCallum and E. Herlt, Exact solutions of Einstein´s field equations, Cambridge University Press (1980)

[54] E. T. Newman and R. Penrose, J. Math. Phys. 3 (1962) 566

[55] S. J. Campbell and J. Wainwright, Gen. Rel. Grav. 8 (1977) 987

[56] L. H. Ryder, Quantum field theory, Cambridge Univ. Press (1985)

[57] C: Lanczos, Am. Scientist 55 (1967) 129

[58] H. Weyl, The theory of groups and quantum mechanics, Dover, NY (1950)

[59] F. A. Kaempfer, Concepts in quantum mechanics, Academic Press, NY (1965)

[60] B.L. Van der Waerden, Group theory and quantum mechanics, Springer-Verlag (1974)

[61] D. Bohm, Perspectives in quantum mechanics, Academic Press, NY (1979)

[62] W. Greiner, Relativistic quantum mechanics, Springer-Verlag (1990)