Relational Relativity

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According to a simple model of inertia a Machianized theory of special and general relativity is presented.

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Introduction

The Famous issues of Newtonian absolute space and time were followed by many constructive critiques of relationalists. The most efficient works of this kind were due to the Ernst Mach, the contemporary physicist and philosopher [1]. Various aspects of Mach's ideas concerning the motion, from Newton's bucket to quantum gravity, have been collected in the proceeding of the conference held at Tübingen (July 1993) for this purpose [2]. Also among recent references the reader is refered to the works of Assis and Ghosh [3,4].

In his critique of Newtonian mechanics (NM), Mach arrived at the following two conclusions:

i) Only the relative motion of a body with respect to other bodies is observable, not motion with regard to absolute space.
ii) The inertial motion of a body is influenced by all the masses in the Universe.

To appreciate fully these two physically pleasant ideas we have applied them in a proposed classical model of inertia [5]. In this model we consider the inertia as a real two-body interaction. For a system of two particles 1 and 2, in an arbitrary non-rotating reference frame S, this force is proportional to the difference of their accelerations with respect to S and to the inertial charges of each particle as follows:

$$\vec{F}_{inertia} = m.c_1\, c_2(\vec{a}_1 - \vec{a}_2) \tag{1}$$

Here m is inertial coupling constant, c_1 and c_2 represent the inertial charges of particles 1 and 2 respectively and $\vec{a}_1$ and $\vec{a}_2$ are their acceleration with respect to S. In a system consisting of N particles the total force imposed on the i th particle is:

$$\vec{F}_i = m.c_i \sum_{j=1}^{N} c_j(\vec{a}_i - \vec{a}_j) \tag{2}$$

where the index i refers to the particle i and summation is over all particles. By definition in the real world the inertial charge and the Newtonian inertial mass of a particle are related as follows:

$$m_i = m.c_i \sum_{j=1}^{all} c_j \tag{3}$$

Summation is taken over all particles in the Universe. This means that the inertial mass of each particle (say labelled j) depends on its own feature (c_j) and a global effect of all particles in the world $(\sum_{j=1}^{all} c_j)$. This may be considered as a simple formulation of Mach's idea concerning inertia. Since local inhomogenities have no observed effect on the inertial mass then it is accepted that the inertial mass is determined by the global structure of the Universe and this is exactly expressed by Eq. (3). Rewriting Eq. (2) in terms of the inertial masses yields a modified form of the Newton's second law, *i.e.* ,

$$\vec{F}_i = m_i \left(\vec{a}_i - \frac{\sum_{j=1}^{all} m_j \vec{a}_j}{\sum_{j=1}^{all} m_j} \right). \tag{4}$$

As it is evident these equations are invariant under a more general transformation than Galilean's. These transformations may be called generalized Galilean transformations with the form:

$$\begin{cases} t' = t \\ \vec{a}' = \vec{a} - \vec{b} \\ \vec{u}' = \vec{u} - t\vec{b} - \vec{v} \\ \vec{x}' = \vec{x} - \frac{1}{2} t^2 \vec{b} - t\vec{v} + \vec{x}_0 \end{cases} \tag{5}$$

where $\vec{b}$, $\vec{v}$ and $\vec{x}_0$ are constant acceleration, velocity and position of reference frame S' with respect to S at $t = 0$ respectively

Eq. (4) satisfies full Machian aspects The one, which is of interest is that, the so-called absolute space is just the frame attached to the center of mass of the Universe in which the Newtonian second law, $\vec{F}_i = m_i\vec{a}_i$ is recovered. The main feature of this model from a Machian point of view is its relational nature, so that the presence of each particle in the Universe and its location relative to the others determine the inertial reference frames Eqs. (2),(4) also satisfy Newton's third law automatically For a two-particle system we have $\vec{F}_1 = -\vec{F}_2$ and for a system with N particles they make $\sum_{i=1}^{N} \vec{F}_i = 0$.

We may also extend this model to gravity. Equivalence principle here means that the source of inertia and gravitation is the same. Let us define the gravitational force between two particles of inertial charges c_1 and c_2 as:

$$|\vec{F}_G| = \frac{m^2 . c_1 . c_2}{|\vec{r}_{12}|^2} \tag{6}$$

Then we can express gravitational constant G in terms of inertial charges c_i;

$$G = \left(\sum_{j=1}^{all} c_j \right)^{-2} \tag{7}$$

or

$$G = \frac{m}{\sum_{j=1}^{all} m_j} . \tag{8}$$

Eqs. (7) , (8) show that G as a global effect is resulted from all inertial charges and we may infer that $\sum_{j=1}^{all} c_j$ is finite. According to the Mach's ideas the so-called physical constants (including G) should be determined from global features of the Universe Thus Eqs. (7) and (8) reveal the very feature of a good Machian model

The Lagrangian function from which Eq.(4) may be extracted is simply obtained by following the canonical procedure of D'Alembert's principle Starting from Eq. (4) and restricting ourselves to systems for which the virtual work of the forces of constraint vanishes we obtain

$$\sum_i \left[\vec{F}_i - m_i \left(\vec{a}_i - \frac{\sum_{j=1}^{all} m_j \vec{a}_j}{\sum_{j=1}^{all} m_j} \right) \right] \cdot d \quad \vec{r}_i = 0 , \tag{9}$$

which is the new form of D'Alembert principle. Here $d \quad \vec{r}_i$ are infinitesimal changes of coordinates as the result of virtual displacement of the system This leads to the result just like the one in ordinary NM except that the kinetic energy T that is equal to $\sum_i \frac{1}{2} m_i v_i^2$ should be replaced by

$$T = \sum \tfrac{1}{2} m_i v_i^2 - \frac{\left[\sum_i m_i v_i\right]^2}{2\sum_i m_i}$$
$$= \tfrac{1}{4} \sum_i \sum_j m_i m_j \frac{\left(\vec{v}_i - \vec{v}_j\right)^2}{\sum_k m_k} \tag{10}$$

Indeed the difference is the second term in the first row and is just the kinetic energy of the center of mass, which is cancelled out in this model. This is well justified when is applied in cosmology. Where we are dealing with the whole Universe, motion and kinetic energy of its center of mass have no physical meaning.

The new form of T as a function of the magnitude of relative velocities of particles has a scalar invariant manner. Then as another advantage in this model the Lagrangian and Hamitonian of a system are scalar invariants from point of view of a nonrotating observer

$$L = \tfrac{1}{4} \sum_i \sum_j m_i m_j \frac{\left(\vec{v}_i - \vec{v}_j\right)^2}{\sum_k m_k} - V(r_{ij}) \tag{11}$$

It is noticeable that in a different way to obtain a relational NM Eq.(11) has been proposed by Lynden-Bell [6,7].

We may summarize the Machian features of this model as follows

1. The relational nature of this model is so that by considering relational distances there is no need to assume absolute space or inertial frame. Indeed the so-called inertial frame is the frame attached to the center of mass of the Universe. Then existence of each particle and its location with respect to others determine the inertial frames

2. Inertial mass of each particle depends on its own inertial charge and the sum of inertial charges of all particles in the world It is not a natural constant, and may change whenever the total inertial charge of the world undergoes any change (*e.g.* in pair production era).
3. Gravitational constant G is related to the sum of all inertial charges existing in the Universe and as a global effect each individual particle shares in its construction. Just like inertial mass, this may be changed whenever the total inertial charge of the world faces with changes.
4. The concept of energy in this model is independent of measuring reference frame and it is an invariant scalar quantity.
5. For an empty universe it does predict no structure.

Collection of these features in the above model provides us a suitable guide to continue and achieve a modified theory of relativity, *i.e.* a theory of relativity without any non-Machian shortcoming, what we may call as relational relativity (RR). As a first step toward RR it is convenient to begin with special relativity (SR).

Relational Special Relativity

At the beginning it should be noticed that according to the Eqs. (7) and (8) it is possible to assume a world without inertia *via* vanishing the coupling constant m, but the assumption of a world without gravitation is physically impossible Then the subject of special relativity because of its ignorance of gravitation is under question and cannot be considered as a global theory from a Machian standpoint. In spite of this we try to present a relational special theory of relativity.

Although Michelson-Morley experiment rejects the concept of ether but SR still is based on the same assumption of the existence of absolute space and preference of inertial frames as NM. In a relational approach we may remove the need for absolute space in SR. To do this task some preliminary remarks should be mentioned.

In NM the Lagrangian of a free particle is just the kinetic energy and its action is

$$S = \int dt(\tfrac{1}{2} m\dot{x}^2) \qquad (12)$$

where $\dot{x}$ is the velocity of the particle with mass m. In SR this is changed to the following form

$$S = -m\int dt(1-\dot{x}^2)^{\frac{1}{2}} = -m\int ds \qquad (13)$$

so that in low velocity limit $(\dot{x} << 1)$ the equation of motion returns to the Newtonian form Other form of this relation in terms of space-time metric is

$$S = -m\int dt\left(g_{mn} \frac{dx^m}{dt}\frac{dx^n}{dt}\right)^{\frac{1}{2}}. \qquad (14)$$

That is the Lagrangian is as follows

$$L = -m\left(g_{mn}\dot{x}^m\dot{x}^n\right)^{\frac{1}{2}} \qquad (15)$$

where $g_{mn} = h_{mn}$ *i.e.*, just the Minkowski metric

According to the definition of canonical momentum we have

$$p_a = \frac{\partial L}{\partial \dot{x}^a} = -\frac{mh_{am}\dot{x}^m}{\left(h_{mn}\dot{x}^m\dot{x}^n\right)^{\frac{1}{2}}} = -mh_{am}u^m \qquad (16)$$

where by definition $\dfrac{dx^m}{ds} = u^m$.

Then the equation of motion has the form

$$m\dot{u}_a = 0. \tag{17}$$

For a system of N particles with masses m_a, $a = 1,2,...,N$ the action is:

$$S = -\sum_{a=1}^{N} m_a \int \left(h_{mn} \frac{dx_a^m}{dp} \frac{dx_a^n}{dp} \right)^{\frac{1}{2}} dp \tag{18}$$

where p is an affine parameter and the Lagrangian is:

$$L = -\sum_{a=1}^{N} m_a \left(h_{mn} \frac{dx_a^m}{dp} \frac{dx_a^n}{dp} \right)^{\frac{1}{2}} dp \tag{19}$$

We should add two other primary remarks about geometrical and physical points. With physical point we mean a point mass but a geometrical point need not contain any matter. We should insist in this fact that a distance measurement is only made between two physical points. So in presenting the line element definition instead of measuring the distance of physical points with respect to an arbitrary origin we should define it in terms of the distance between physical points (or physically significant points *e.g.*, center of mass of a system). Certainly this definition has higher Machian (Relational) validity. Now from this point of view let us define the line element ds_a^2 for a noninteracting N particle system as:

$$ds_a^2 = h_{mn} \left(dx_a^m - \frac{\sum_b m_b dx_b^m}{\sum_b m_b} \right) \left(dx_a^n - \frac{\sum_b m_b dx_b^n}{\sum_b m_b} \right) \tag{20}$$

where index (a) refers to the particle labeled (a). Then the related action and Lagrangian are as follows:

$$S = -\sum_a m_a \int \left[h_{mn} \left(\frac{dx_a^m}{dp} - \frac{\sum_b m_b \frac{dx_b^m}{dp}}{\sum_b m_b} \right) \left(\frac{dx_a^n}{dp} - \frac{\sum_b m_b \frac{dx_b^n}{dp}}{\sum_b m_b} \right) \right]^{\frac{1}{2}} dt \quad (21)$$

$$L = -\sum_a m_a \left[h_{mn} \left(\frac{dx_a^m}{dp} - \frac{dx_{cm}^m}{dp} \right) \left(\frac{dx_a^n}{dp} - \frac{dx_{cm}^n}{dp} \right) \right]^{\frac{1}{2}}. \quad (22)$$

The canonical momentum of the kth particle is

$$\begin{aligned}(p_k)_a &= \frac{\partial L}{\partial \frac{dx_k^a}{dp}} = -m_k h_{an} \frac{\left(\frac{dx_k^n}{dp} - \frac{dx_{cm}^n}{dp} \right)}{\frac{ds_k}{dp}} \\ &= -m_k h_{an} \left(u_k^n - u_{cm}^n \right) \quad (23) \\ &= -m_k \left((u_k)_a - (u_{cm})_a \right)\end{aligned}$$

Since $\frac{\partial L}{\partial x_k^a} = 0$ and $\frac{d(p_k)_a}{dp} = -m_k h_{an} \left(\frac{du_k^n}{dp} - \frac{du_{cm}^n}{dp} \right)$, then the equation of motion of the kth particle is

$$\frac{d\vec{u}_k}{dp} - \frac{d\vec{u}_{cm}}{dp} = 0 \quad (24)$$

which is just the same as the modified form (4) in the Newtonian limit.

The Lagrangian (22) is written without any coordination with respect to a priori fixed virtual absolute space and these are particles by their own relative locations that determine it. This is free from those non-Machian aspects suffering the standard SR. So we may call the relativistic theory based on this Lagrangian as relational special relativity

Relational General relativity

It seems the same approach may be followed to obtain the relational GR. But this is not so straightforward. Because to extrapolate this result to GR, *i.e.*, to change the Minkowskian flat space-time (h_{mn}) into the Riemannian curved space-time (g_{mn}), care should be taken of dealing with vector quantities. Summation of the vectors in this case needs parallel transportation of them, which in turn requires defining the path of transportation for each of them. To achieve a relational theory of GR it requires choosing another strategy with some different approach as follows.

Initially we remark the center of mass (CM) concept in NM With the help of this concept in the Euclidian space NM of a single particle can be extrapolated and be applied to a system with N particles. The classical meaning of CM losses its uniqueness when enters in the realm of relativity so that different observers find different points as CM of a given system. The important point worthy to notice about CM is its dual character from a Machian point of view, so that despite of its great value as a technical tool to present the relational motion, on the other hand as a point in which total mass of the system is located and its motion is to be considered, it is quite anti-Machian. A single particle has no motion and no inertia.

Turning back to the NM we may define the center of inertial charge (CI). By definition

$$X_{CI}^{m} = \frac{\sum_i \sum_j c_i c_j \left(x_i^m + x_j^m \right)}{2\left(\sum_j c_j \right)^2} = \frac{\sum_i m_i x_i^m}{\sum_i m_i} = X_{CM}^{m} \tag{25}$$

where X_{CM}^{m} and X_{CI}^{m} are coordinates of CM and CI respectively. As it is evident the concept of CI has also a mutually relational content between particles. Now it is easy to show that the result (4) may be obtained with the help of Lagrangian formalism in NM and imposing the following condition on CI

$$d \quad X_{CI}^{m} \equiv 0 . \tag{26}$$

Because

$$d \quad X_{CI}^{m} = \sum_i m_i d \quad x_i^m = 0 , \tag{27}$$

and imposing this by using the method of undetermined Lagrangian multipliers in variations of the action of a system with N noninteracting particles , yields

$$d \quad I = \sum_n \int dt \left(m_n \ddot{x}_n^m + f \quad m_n \right) d \quad x_n^m \equiv 0 \tag{28}$$

where the coefficient *f* is determined as follows

$$f = -\frac{\sum_n m_n \ddot{x}_n^m}{\sum_n m_n} . \tag{29}$$

Thus the equation of motion (4) is obtained. Also with consideration of the condition (27) in variation of the action (18) in special relativity the result (22) may be obtained.

To remove the Machian objection to the concept of CM the following condition as a Machian condition may be imposed to the variations of the dynamical variables x_n^m of the system Let us first define X^m as

$$X^m \equiv \sum_n m_n x_n^m . \tag{30}$$

Of course X^m is not a vector quantity and depend to the chosen reference frame. Let us denote its variations with $d \quad X^m$:

$$d \quad X^m \equiv \sum_n m_n d \quad x_n^m \tag{31}$$

Here $d \quad X^m$ is not vector while $d \quad x_n^m$ are vectors. Similarly $d \quad X_m$ is defined as follows

$$d \quad X_m \equiv \sum_n m_n g_{ml}(x_n) d \quad x_n^l \tag{32}$$

Now as a Machian principle we postulate that always $d \quad X_m$ vanishes (lower index is chosen only for convenience). This means that variations of dynamical variables x_n^m are under the following condition

$$\sum_n m_n g_{ml}(x_n) d \quad x_n^l = 0 \tag{33}$$

Despite of this fact that (33) is not a covariant condition, we can make the best use of it to find at least a clue for the geodesic equations in GR.

Matter action for a system consisting of n particles with masses m_n is given by the following form in GR;

$$I = \sum_n m_n \int dp \left(g_{mn}(x_n(p)) \frac{dx_n^m(p)}{dp} \frac{dx_n^n(p)}{dp} \right)^{\frac{1}{2}} \tag{34}$$

where p is some quantity that simultaneously parameterizes all the space-time trajectories of the various particles.

Variation of the action (34) due to an infinitesimal variation in the dynamical variables $x^m \rightarrow x^m(p) + d \quad x^m(p)$ is given by

$$d \quad I = \frac{1}{2} \sum_n m_n \int dp \left[g_{mn}(x_n(p)) \frac{dx_n^m(p)}{dp} \frac{dx_n^n(p)}{dp} \right]^{-\frac{1}{2}}$$

$$\times \left\{ 2 g_{mn}(x_n(p)) \frac{dx_n^m(p)}{dp} \frac{dx_n^n(p)}{dp} \right. \tag{35}$$

$$\left. + \left(\frac{\partial g_{mn}(x)}{\partial x^l} \right)_{x = x_n(p)} \frac{dx_n^m(p)}{dp} \frac{dx_n^n(p)}{dp} d \quad x_n^l(p) \right\}$$

It is convenient to change variables of integration (35) from p to the t_n (the proper time of the particle n) defined by

$$dt_n \equiv \left(g_{mn} dx_n^m dx_n^n \right)^{\frac{1}{2}} \tag{36}$$

So the integral in (35) may be written in a simpler form

$$d \quad I = \frac{1}{2} \sum_n m_n \int dt_n \left\{ 2 g_{ml}(x_n) \frac{dx_n^m}{dt_n} \frac{dd \quad x_n^l}{dt_n} \right.$$

$$+\frac{\partial g_{mn}(x_n)}{\partial x_n^l}\frac{dx_n^m}{dt_n}\frac{dx_n^n}{dt_n}d \quad x_n^l \Bigg\} \tag{37}$$

Finally integration by parts of the first term in (37) with the condition that $d \quad x^m(t_n)$ vanishes on the boundaries of integration yields that

$$d \quad I=\sum_n \int dt_n g_{ml}(x_n)\left\{m_n\left(\frac{d^2x_n^m}{dt_n^2}+\Gamma^m_{rs}\frac{dx_n^r}{dt_n}\frac{dx_n^s}{dt_n}\right)\right\}d \quad x_n^l \tag{38}$$

where Γ^m_{rs} are the second type Christoffel symbols. Then according to the principle of stationary action, $d \quad I$ vanishes for general variations in the dynamical variables $d \quad x_n^l$ if and only if the dynamical variables obey the geodesic equations

$$\frac{d^2x_n^m}{dt_n^2}+\Gamma^m_{rs}\frac{dx_n^r}{dt_n}\frac{dx_n^s}{dt_n}=0 \tag{39}$$

Now we repeat the above standard process with consideration of the Machian condition (33) to achieve the equations of motion To impose the mentioned condition with the method of undetermined Lagrangian multipliers it is enough only to add the following term to the variations of the action (34)

$$\int dpf^m\sum_n m_n g_{ml}(x_n)d \quad x_n^l \tag{40}$$

where f^ms are undetermined coefficients and just as in (34) parameter p is an arbitrary quantity which simultaneously parameterizes the space-time trajectories of different particles Then we have

$$d \quad I=\sum_n \int dp \left\{ m_n g_{ml}(x_n) \left[\frac{\partial p}{\partial t_n} \left(\frac{d^2 x_n^m}{dp^2} + \Gamma^m_{rs} \frac{dx_n^r}{dp} \frac{dx_n^s}{dp} \right) + f^m \right] \right\} d \quad x_n^l = 0 \tag{41}$$

With f^m determined as:

$$f^m = -\frac{\sum_n m_n \frac{\partial p}{\partial t_n} \left(\frac{d^2 x_n^m}{dp^2} + \Gamma^m_{rs} \frac{dx_n^r}{dp} \frac{dx_n^s}{dp} \right)}{\sum_n m_n} \tag{42}$$

Because of the mean operation over all particles, f^m is a global quantity. Therefore by inserting the value of f^m the Machianized form or the relational form of the geodesic equations of motion are derived as follows

$$\frac{d^2 x_n^m}{dp^2} + \Gamma^m_{rs} \frac{dx_n^r}{dp} \frac{dx_n^s}{dp} - \frac{\sum_j m_j \frac{\partial t_n}{\partial t_j} \frac{d^2 x_n^m}{dp^2}}{\sum_j m_j} - \frac{\sum_j m_j \frac{\partial t_n}{\partial t_j} \Gamma^m_{rs} \frac{dx_j^r}{dp} \frac{dx_j^s}{dp}}{\sum_j m_j} = 0 \tag{43}$$

It reveals that in the weak field limit the equations (43) correspond with the Newtonian one, because the Christoffel symbols vanish and parameters t_n in this limit are all the same and are equal to t then Eq.(43) reduces to the modified Newtonian form Eq.(4).

Now according to the relational result (43) we may propose the covariant form of the geodesic equations as follows

$$\frac{d^2 x_n^m}{dp^2} + \Gamma^m_{rs}\frac{dx_n^r}{dp}\frac{dx_n^s}{dp} - \frac{\sum_j m_j \frac{\partial t_n}{\partial t_j} U_{x_j}^{x_n}\left(\frac{d^2 x_j^m}{dp^2}+\Gamma^m_{ab}\frac{dx_j^a}{dp}\frac{dx_j^b}{dp}\right)}{\sum_j m_j} = 0 \qquad (44)$$

Where $U_{x_j}^{x_n}$ is the parallel transportation operator from the location of the jth particle to the location of the nth one.

Remarks

We are now staying at a standpoint that may return to the famous question that ``whether the formalism of general relativity and the Einstein equations are perfectly Machian?'' and have a strictly positive answer to it. Checking the Machian (or anti-Machian) aspects of GR we notice that;

1. By now, in front of the basic question that why the Einstein field equations have nontrivial solution flat space $R_{mn} = 0$ for empty universe, we had to resort to the boundary condition reasons. Hereafter, with what we have find about inertia it is seen that the Einstein field equations $\frac{c^4}{8p\ G} R_{mn} = \left(T_{mn} - \frac{1}{2} g_{mn} T\right)$ predict $0 = 0$ (instead of $R_{mn} = 0$) for empty universe. For assuming vacuum $T,\ T_{mn} = 0$ makes the RHS of the field equations to be equal zero and on the other side the coupling constant appears on the LHS as G^{-1}, which in turn according to the Eqs. (7) and (8) depends on the existence of all particles in

the universe, $G \propto \frac{1}{\sum_i m_i}$, so for the empty universe $\sum_i m_i = 0$ and thus the field equations yield $0 = 0$, which is a perfectly Machian result.

2. For a world with a single particle, although the field equations based on the presented model of inertia predict a solution that is independent of inertial charge and merely depending to the coupling constant m. But for its geodesic equation the Eqs. (43) and (44) yield the result $0 = 0$ that means denying any motion for a single particle, an ideal result from a Machian point of view.

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