The Wilsons' Experiment

Pierre Hillion Institut Henri Poincaré 86 Bis Route de Croissy, 78110 le Vesinet, France

Many years ago, the difference of potential between the inner and outer surfaces of a hollow, circular, dielectric cylinder rotating in a constant magnetic field parallel to its axis, was measured by the Wilsons. There is still some controversy whether their measurement is in agreement or not with relativity. In this work, for a cylinder rotating with a small angular velocity, we make a complete and fully covariant analysis (the v^2c^{-2} terms are not neglected) of this experiment. We confirm the previous theoretical results obtained in a less rigorous and non-covariant way.

1 Introduction

The electrodynamics of moving objects has been in the past the subject of many experimental and theoretical works. The magnetic effect of a moving charge was first observed by Rowland [I] and that due to a moving dielectric by Röntgen [2]. These effects were confirmed and placed on a quantitative basis by Eichenwald [3] whose careful experiments were analysed by Jones[4]. Then, on a suggestion from Einstein and Laub [5], the Wilsons [6] performed an experiment which is the counterpart to Eichenwald's. A dielectric cylinder was made to rotate between the plates of a condenser in a magnetic field parallel to the plates. A difference of potential was observed between the plates.

A simple way to explain this result [7] is to replace rotation by a rectilinear motion, parallel to the plates but at right angles to the magnetic field. Then, at low velocity, that is neglecting the v^2c^{-2} terms a simple relativistic calculation gives [8] for the difference of potential ΔV between the plates

$$\Delta V = \mu \left(1 - \frac{1}{\mu \varepsilon} \right) v l H_0 \tag{1}$$

In this expression H_0 is the constant magnetic field, l the distance between the plates, ε,μ the characteristics of the dielectric material, v the constant velocity parallel to the plates.

So, if a,b, are the outer and inner radius of the dielectric cylinder, Ω its uniform angular velocity of rotation, one has approximately: l = a - b, $v = \Omega r = \Omega(a + b)/2$ and Eq.(1) becomes

$$\Delta V = \mu M \Omega H_0 \frac{a^2 - b^2}{2}, \quad M = 1 - \frac{1}{\varepsilon \mu}$$
 (2)

In the Wilsons' experiment $\varepsilon = 6$ $\mu = 3$, so M = 0.944 and they found M = 0.96 while the Lorentz theory of electrons gave $M = 1 - 1/\varepsilon = 0.83$. Taking into account the poor accuracy of the Wilsons' measurements, it is generally considered that their result confirms relativity [7,8].

But relation (2) may also be obtained in a less simplistic way by using electrodynamics in rotating media. In this case, one uses a corotating frame K', the most suitable being the Frenet-Serret tetrad [9,10] which plays the role of an instantaneous inertial frame. Then, in K' an holonomic de-

scription of the electromagnetic field is obtained by assuming that it rotates with K' and Maxwell's equations are:

$$(-g)^{1/2} \partial_j \left[(-g)^{1/2} F^{jk} \right] = J^k, \quad \partial_j \left[(-g)^{1/2} G^{jk} \right] = 0$$
 (3)

The Latin indices take the value 0,1,2,3, and the summation convention is used, F^{ik} , G^{jk} , are the electromagnetic tensors, J^k the current density and g the determinant of the space-time metric tensor in K'. For the nonholonomic description, Maxwell's equations are

$$F_{lj}^{jk} = J^k, \quad \Gamma_{lj}^{jk} = 0 \tag{4}$$

with for any tensor T^{jk} (Γ_m^{jk} are the Christoffel symbols)

$$T_{lm}^{jk} = \partial_m T^{jk} + \Gamma_{lm}^j T^{lk} + \Gamma_{lm}^k T_j^j \tag{5}$$

Unfortunately all the works on rotating media use a Galilean description of rotations with azimuthal velocity $v = \Omega r$ and it has been proved recently [11] that with this description, Eq. (1) and Eq.(2) differ in the Galilean corotating frame and no one has the inertial form; that is, the covariance of Maxwell's equations is broken down. This result challenges the relation (2) and it has been argued recently [12,13] that there is an apparent conflict between theory and the Wilsons' experiment.

This suggests that one should work in rotating media with a largely ignored relativistic description of rotations put forward by Trocheris [14] and Takeno [15] many years ago, specially since it was also proved [11] that with this description both equations (3), (4), coincide with the inertial form of Maxwell's equations, so that the full covariance of electromagnetism is restored. We make here a rigourous analysis of the Wilsons' experiment in terms of the relativistic Trocheris-Takeno description of rotations.

2. Basic equations

We consider a semi-infinite circular dielectric cylinder rotating with a uniform angular velocity Ω around its axis parallel to a static magnetic field $H_0\mathbf{i}_z$ in which \mathbf{i}_z is a unit vector in the z direction. Since this problem is axisymmetric, Maxwell's equations with the cylindrical coordinates r, ϕ z reduce to

$$\partial_z E_r - \partial_r E_z = 0, \quad \partial_z H_r - \partial_r H_z = 0$$
 (6a)

$$r^{-1}\partial_r(rD_r) + \partial_z D_z = 0, \quad r^{-1}\partial_r(rB_r) + \partial_z B_z = 0 \tag{6b}$$

corresponding respectively to the curl and divergence equations; **E**,**B**,**D**,**H**, are the usual components of the electromagnetic field. Outside the cylinder these equations are supplemented by the constitutive relations $\mathbf{D} = \varepsilon_0 \mathbf{E}$, $\mathbf{B} = \mu_0 \mathbf{H}$ and we have to get the form of the constitutive relations inside the cylinder.

As previously said, we use the Trocheris-Takeno description [14,15] of rotations between the laboratory frame K and the corotating Frenet-Serret frame K in which the dielectric is at rest

$$r = r', \ z = z', \ \phi = \cosh\beta\phi' - \sinh\beta\frac{x'_0}{r}, \ x_0 = -\sinh\beta r\phi' + \cosh\beta x'_0$$
 (7)

with $\beta = \Omega r/c$, $x_0 = ct$, $x_0' = ct'$. The small velocity approximation of (7) consistent with relativity is with $\gamma = 1 + \beta^2/2$

$$r = r', \ z = z', \ \phi = \gamma \left(\phi' - \Omega r'\right), \ ct = \gamma \left(ct' - \frac{\Omega r^2 \phi'}{c}\right)$$
 (7a)

which is to the order $0(\beta^3)$. Note that even neglecting the β^2 -terms that is making $\gamma = 1$ does not yield the Galilean rotations: r = r', z = z'. $\phi = \phi' - \Omega t$, t = t'. With (7) the velocity $\mathbf{v}(|\mathbf{v}| \le c)$ is (\mathbf{i}_{ϕ} is a unit azimuthal vector)

$$\mathbf{v} = cs\mathbf{i}_{\phi} \ , \ s = \tanh \beta \tag{8}$$

In the Frenet-Serret frame, the constitutive relations are $\mathbf{D}' = \varepsilon \mathbf{E}'$, $\mathbf{B}' = \mu \mathbf{H}'$ so that using the well known transformations [16] of the electromagnetic field between two inertial frames (K' is an instantaneous inertial frame), we get in K

$$\mathbf{D} + c^{-1}\mathbf{v} \wedge \mathbf{H} = \varepsilon \left(\mathbf{E} + c^{-1}\mathbf{v} \wedge \mathbf{B} \right)$$

$$\mathbf{B} - c^{-1}\mathbf{v} \wedge \mathbf{E} = \mu \left(\mathbf{H} - c^{-1}\mathbf{v} \wedge \mathbf{D} \right)$$
(9)

Substituting (8) into (9) and using the quantities

$$a(s) = (1 - s^2)(1 - n^2 s^2)^{-1}, \quad b(s) = m^2 s (1 - n^2 s^2)^{-1}, \quad n^2 = \varepsilon \mu, \quad m^2 = n^2 - 1$$
 (10)

A simple calculation gives

$$D_r = \varepsilon a(s) E_r + b(s) H_z, \quad D_z = \varepsilon a(s) E_z - b(s) H_r$$

$$B_r = \mu a(s) H_r - b(s) E_z, \quad B_z = \mu a(s) H_z + b(s) E_r$$
(11)

which complete Maxwell's equations inside the cylinder.

So, taking into account the constitutive relations, one has to solve Maxwell's equations inside and outside the cylinder and to match the solutions on its surface to ensure the continuity of E_z , H_z and D_r , B_r . One has in addition to satisfy the condition at infinity $\mathbf{H} = H_0 \mathbf{i}_z$. Now it is clear that in the Wilsons' experiment, the angular velocity could not be very large, so we shall tackle this problem in the framework of small velocities.

Remark: It is important to note that unlike the Frenet-Serret tetrad, the Galilean corotating frame is not an instantaneous inertial frame. So it is unjustified to work with the transformations (9) to tackle the constitutive relations: a point also stressed in [12]. But as proved in other work [11], the anholonomic equations are no more covariant in this case.

3. Low-velocity approximation

To make approximations easier, we introduce the dimensionless parameter $\alpha = \Omega a/c$ where a is the outer radius of the cylinder, then $\beta = \alpha r/a < a$. As previously said, the lowest approximation consistent with relativity is $0(\alpha^3)$ because $\tanh \beta = \beta + 0(\beta^3)$. But it is interesting to start with the $0(\alpha^2)$ approximation and we shall prove that for the Wilsons' experiment both approximations give the same result.

We get from (10) $s = \alpha r/a + 0(\alpha^3)$ and

$$a(s) = 1 + 0(\alpha^2) = 1 + \alpha^2 m^2 r^2 a^{-2} + 0(\alpha^3), \quad b(s) = \alpha m^2 r^{-1} + 0(\alpha^{2,3})$$
(12)

Now the solutions of the two curl equations (6a) can be obtained in terms of two scalar potentials χ, ψ :

$$H_r = \partial_r \chi; \ H_z = \partial_z \chi; \ E_r = \alpha \partial_r \psi, \ E_z = \alpha \partial_z \psi$$
 (13)

The introduction of α in the definition of **E** is consistent with the fact that the electric field is zero for a stationary cylinder. Substituting (12) and (13) into (11) gives

$$B_r = \mu \partial_r \chi + 0(\alpha^2) = \mu \left(1 + \alpha^2 m^2 r^2 a^{-2}\right) \partial_r \chi - \alpha^2 m^2 r a^{-1} \partial_z \psi + 0(\alpha^3)$$

$$B_z = \mu \partial_z \chi + 0(\alpha^2) = \mu \left(1 + \alpha^2 m^2 r^2 a^{-2}\right) \partial_z \chi - \alpha^2 m^2 r a^{-1} \partial_r \psi + 0(\alpha^3)$$
(14a)

$$D_{r} = \alpha \left(\varepsilon \partial_{r} \psi + m^{2} r a^{-1} \partial_{z} \chi \right) + 0 \left(\alpha^{2,3} \right)$$

$$D_{z} = \alpha \left(\varepsilon \partial_{z} \psi + m^{2} r a^{-1} \partial_{r} \chi \right) + 0 \left(\alpha^{2,3} \right)$$
(14b)

So, **D** has the same expression to the orders $0(\alpha^2)$ and $0(\alpha^3)$.

3.1 $O(\alpha^2)$ -approximation of the magnetic potential

In this section, we work in the framework of the $0(\alpha^2)$ approximation, so we suppress this symbol. Substituting (14a) into the divergence equation (6b) gives

$$\Delta \chi = 0, \quad \Delta = r^{-1} \partial_r (r \partial_r) + \partial_z^2$$
 (15)

So, the magnetic potential satisfies the same Laplace equation inside and outside the cylinder. Taking in to account the condition at infinity $\mathbf{H} = H_0 \mathbf{i}_z$ we may write the solutions of (15)

$$\chi_{out} = H_0 z + \int_0^\infty e^{-kz} Y_0(kr) f(k) dk$$
 (16a)

$$\chi_{in} = \int_{0}^{\infty} e^{-kz} J_0(kr) g(k) dk$$
 (16b)

 J_0 Y_0 , are the Bessel functions of the first and second kind of order zero. The functions f(k) and g(k) to be determined by the boundary conditions on the surface r = a of the cylinder are supposed to ensure uniform convergence so that one may take the derivatives under the integral sign. In addition, we write

$$H_0 z = 2H_0 \int_0^\infty e^{-kz} J_0(kr) \delta'(k) dk$$
 (17)

where $\delta'(k)$ is the derivative of the Dirac distribution. We recall that

$$\int_{0}^{\infty} f(x)\delta'(x)dx = -\frac{f'(0)}{2}$$
(18)

and it is easy to check the correctness of (17) since $J_0(0) = 1$, $J'_0(0) = -J_1(0) = 0$. Then, using (16) and (17), the continuity of H_z on r = a implies

$$2H_0J_0(ka)\partial'(k) + f(k)Y_0(ka) = g(k)J_0(ka)$$
(19a)

while using the relations $J'_0 = -J_1$, $Y'_0 = -Y_1$ we get from the continuity of B_r with $\mu_r = \mu/\mu_0$

$$2H_0J_1(ka)\partial'(k) + f(k)Y_1(ka) = \mu_r g(k)J_1(ka)$$
(19b)

From Eqs.(19) a simple calculation gives with $w(ka) = [Y_1J_0 - \mu_r Y_0J_1]_{ka}$

$$g(k) = 2H_0 w^{-1} (ka) [Y_1 J_0 - Y_0 J_1]_{ka} \partial'(k) = -4H_0 w^{-1} (ka) \frac{\partial'(k)}{\pi ka}$$
 (20a)

$$f(k) = 2(\mu_r - 1)H_0 w^{-1}(ka)J_0(ka)J_1(ka)\partial'(k)$$
(20b)

In (20a) we used the fact that the numerator of g(k) is the Wronskian of the Bessel functions Y and J. Substituting (20) into (16) would supply the magnetic potential but according to (18) we need the expressions of f(k) and g(k) in the neighborhood of k = 0. We use the following approximations [17] in which γ is the Euler constant

$$J_{0}(x) = 1 + 0(x^{2}), \quad J_{1} = \frac{x}{2} + 0(x^{3})$$

$$Y_{0}(x) = 2\pi^{-1}\left(\gamma + \frac{\log x}{2}\right) + \left(x^{2}\log x\right), \quad Y_{1}(x) = -\frac{2}{\pi x} + x\pi^{-1}\left(g + \frac{\log x}{2}\right) + 0(x^{2}\log x)$$
(21a)

so that:

$$w(ka) = -\frac{2}{\pi ka} \left[1 + 0(k^2 a^2 \log ka) \right]$$
 (21b)

Taking into account (21), the expressions (20) become

$$f(k) = -\pi(\mu_r - 1)H_0k^2a^2\delta'(k) + 0(k^2a^2\log ka)$$
(22)

Substituting (22) into (16) and using (18) gives at once $\chi_{in} = \chi_{out} = H_0 z$. So to the order $O(\alpha^2)$ the magnetic potential is the same inside and outside the cylinder and according to (13)

$$H_{r,in} = H_{r,out} = 0, \quad H_{z,in} = H_{z,out} = H_0$$
 (23)

3.2 $O(\alpha^2)$ -approximation of the electric potential

Substituting (14b) into the divergence equation (6b) and using (23) show that the electric potential inside the cylinder satisfies the Poisson equation

$$\Delta \psi_{in} = -2m^2 \left(a\varepsilon\right)^{-1} H_0 \tag{24}$$

with the solution

$$\psi_{in} = \int_{0}^{\infty} e^{-kz} \left[-m^2 r^2 \left(a\varepsilon \right)^{-1} H_0 \delta(k) + g(k) J_0(kr) \right] dk$$
 (25a)

and since outside the cylinder $\Delta \psi_{out} = 0$

$$\psi_{out} = \int_{0}^{\infty} e^{-kz} f(k) Y_0(kr) dk$$
 (25b)

The continuity of E_z for r = a gives

$$-m^2 a \varepsilon^{-1} H_0 \delta(k) + g(k) J_0(ka) = f(k) Y_0(ka)$$
(26)

The continuity of D_r requires more attention. First since $\chi_{in} = H_0 z$ we have according to (17)

$$\partial_z \chi_{in} = 2H_0 \int_0^\infty e^{-kz} J_0(kr) \delta(k) dk$$
 (27)

Then, substituting (25a) and (27) into (14b) gives

$$D_{r,in} = -\alpha \varepsilon k \int_{0}^{\infty} e^{-kz} \Big[2m^{2} r (a\varepsilon k)^{-1} H_{0} \Big\{ 1 - J_{0}(kr) \Big\} \delta(k) + g(k) J_{1}(kr) \Big] dk$$

$$= -\alpha \varepsilon k \int_{0}^{\infty} e^{-kz} g(k) J_{1}(kr) dk$$
(28a)

since the coefficient of $\delta(k)$ is null for k = 0 (f(x) $\delta(x) = f(0)$ $\delta(x)$). Now according to (25b)

$$D_{r,out} = -\alpha \varepsilon_0 k \int_0^\infty e^{-kz} f(k) Y_1(kr) dk$$
 (28b)

and the continuity of D_r for r = a implies with $\varepsilon_r = \varepsilon / \varepsilon_0$

$$\varepsilon_r g(k) J_1(ka) = f(k) Y_1(ka) \tag{29}$$

Using the approximations (21) we get from (26) and (29) with $w_e(ka) = [Y_1J_0 - \varepsilon_r Y_0J_1]_{ka}$

$$f(k) = m^{2} a \varepsilon^{-1} H_{0} w e^{-1} (ka) \delta(k) = 0$$

$$g(k) = m^{2} a \varepsilon^{-1} H_{0} w e^{-1} (ka) Y_{1} (ka) \delta(k) = m^{2} a \varepsilon^{-1} H_{0} \delta(k)$$
(30)

Substituting (30) into (25) gives

$$\psi_{in} = \frac{m^2 H_0 \left(a^2 - r^2\right)}{2a\varepsilon}, \quad \psi_{out} = 0 \tag{31}$$

So, according to (13) and (31) we have to the order $0(\alpha^2)$

$$E_{r,in} = \frac{\Omega m^2 H_0 r}{c \varepsilon}, \quad E_{z,in} = E_{r,out} = E_{z,out} = 0$$
 (32)

Since $m^2 = \varepsilon \mu - 1$ one checks at once that $E_{\varepsilon in}$ is the expression (2).

3.3 $O(\alpha^3)$ -approximation

One sees easily that the $0(\alpha^3)$ approximation supplies the same result. Indeed according to (28) and (30): $D_{r,in} = D_{r,out} = 0$. Then, using (11) and (12):

$$E_{r} = -\frac{b(s)}{\varepsilon a(s)} H_{0}$$

$$= -\frac{\alpha m^{2} r}{\varepsilon a} H_{0} + 0(\alpha^{2,3})$$
(33)

However, it is interesting to prove this result directly. First, since according to (14b) D_r and D_z do not change, the electric potential still satisfies the Poisson equation

$$\varepsilon \Delta \psi + 2m^2 a^{-1} \partial_z \chi = 0 \tag{34}$$

Now substituting the $O(\alpha^3)$ approximation (14a) into the divergence equation (6b) gives

$$\mu \left(1 + \alpha^2 m^2 r^2 a^{-2} \right) \Delta \chi + 2\alpha^2 m^2 r a^{-2} \partial_r \chi - 2\alpha^2 m^2 a^{-1} \partial_z \psi = 0$$
 (35)

Multiplying this equation by $(1 - \alpha^2 m^2 r^2 a^{-2})$ we get to the order $0(\alpha^3)$

$$\mu \chi + 2\alpha^2 m^2 r a^{-2} \partial_r \chi - 2\alpha^2 m^2 a^{-1} \partial_z \psi = 0$$
(36)

We look for the solutions of Eqs.(34) and (36) in the form

$$\chi = \chi^0 + \alpha^2 \chi^1, \quad \psi = \psi^0 + \alpha^2 \psi^1 \tag{37}$$

in which χ^0 , ψ^0 are the $0(\alpha^2)$ approximations of the magnetic and electric potentials satisfying respectively the Laplace equation (15) and the Poisson equation (24). Substituting (37) into (34) and (36) gives for χ ' and ψ^1 the Poisson equations

$$\mu \Delta \chi^{1} + 2m^{2} r a^{-2} \partial_{r} \chi^{0} - 2m^{2} a^{-1} \partial_{z} \psi^{0} = 0$$
 (38a)

$$\varepsilon \Delta \psi^1 + 2m^2 a^{-1} \partial_z \chi^1 = 0 \tag{38b}$$

but according to (23) and (32): $\partial_r \chi^0 = \partial_z \psi^0 = 0$. So Eq.(38a) reduces to $\Delta \chi^1 = 0$ with the trivial solution $\chi^1 = 0$ implying $\Delta \psi^1 = 0$ and finally $\psi^1 = 0$. Consequently, the $O(\alpha^3)$ approximation in this problem does not change the results supplied by the $O(\alpha^2)$ approximation.

4. Conclusion

A careful covariant analysis of the electromagnetic field inside a hollow circular dielectric cylinder rotating with a small uniform angular velocity around a static magnetic field parallel to its axis confirms the expression of the electric field previously obtained by noncovariant methods. So, if there exists a conflict between this result and the measurements made by the Wilsons in 1913, a new experiment should be carried out with better accuracy. A persistent discrepancy would imply that the theory of electrodynamics in rotating media has to be revised.

References

[1] H.A. Rowland, Sitzber Akad Wiss. Berlin 43, 211(1876).

- [2] W.E. Röntgen, Ann. der Phys. 35, 264 (1888).
- [3] A.Eichenwald, Ann.der Phys. 11, 421 (1903).
- [4] D.S.Jones, *The Theory of Electromagnetism* Oxford, Pergamon (1964).
- [5] A. Einstein and J. Laub, *Ann.der Phys.* **26**, 532 (1908).
- [6] M. Wilson and H.A.Wilson, Proc. Roy. Soc. London A 89, 99 (1913).
- [7] W. Pauli, Theory of Relativity, London, Pergamon (1958).
- [8] J.Van Bladel, *Relativity and Engineering*, Berlin, Springer (1984).
- [9] W.M. Irvine, *Physica* **30**, 1160 (1964).
- [10] T.C. Mo, J. Math. Phys. 11, 2589 (1970).
- [11] S. Kichenassamy and R.A. Krikorian, J. Math. Phys. 35, 5726 (1994).
- [12] G.N. Pellegrini and A.R. Swift, Am. J. Phys. 63, 694 (1995).
- [13] M.L. Burrows, Am. J. Phys. 65, 929 (1997).
- [14] M.G. Trocheris, *Philos. Mag.* **40**, 1143 (1949).
- [15] H. Takeno, Prog. Theor. Phys. 7, 367 (1952).
- [16] H. Minkowski, Gottinger Nachrichten 53, 116 (1908).
- [17] M. Abramowitz and I.A. Stegun, Handbook of Mathematical Functions, New York, Dover (1965).

NINTH LOMONOSOV CONFERENCE ON ELEMENTARY PARTICLE PHYSICS

The 9th Lomonosov Conference on Elementary Particle Physics will be held from 20 to 26 September, 1999 at Moscow State University, Moscow, Russia.

The conference is organized by the Interregional Centre for Advanced Studies (Moscow) and the Faculty of Physics of the Moscow State University and supported by the Joint Institute for Nuclear Research (JINR, Dubna), the Institute for Theoretical and Experimental Physics (ITEP, Moscow), the Institute for High Energy Physics (IHEP, Protvino) and the Institute for Nuclear Research (INR, Moscow).

The Lomonosov Conferences bring together about 100 theorists and experimentalists to review the present status and future prospects in elementary particle physics. The program of the 9th Lomonosov Conference will include:

- Electroweak Theory, Tests of Standard Model and Beyond
- Heavy Quark Physics
- Non-Perturbative QCD
- Neutrino Physics
- Astroparticle Physics
- Quantum Gravity Effects

This conference is the last of the series of the Lomonosov Conferences on Elementary Particle Physics in the XX century. As a result, in addition to the Invited Talks (30 min), Session (20 min) and Brief (15 min) Reports there will be a special session of General Talks (50 min) devoted to reviews of the most fundamental results and ideas in elementary particle physics of the century now ending.

Interested persons are kindly asked to send (before April 20, 1999) the title(s) and abstract(s) of the proposed talks to the Conference Secretary Andrey Egorov by e-mail address: ane@srdlan.npi.msu.su

Application forms (which will be used by Organizing Committee for preparation of official invitations for obtaining a visa to visit Russia) for participants are sent by the Organizing Committee upon request by e-mail. Personal data and information required by the application forms should be sent to the Organizing Committee before April 20.

Alexander Studenikin
Department of Theoretical Physics
Moscow State University
119899, Moscow, Russia
phone: (007-095) 939-50-47, 939-31-77

fax: (007-095) 932-88-20

E-mail: studenik@srdlan.npi.msu.su