Arbitrary motion of a viscous incompressible liquid

K.B. Ranger
Department of Mathematics
University of Toronto
Toronto, Ontario M5S 3G3
Canada

It is shown by a constructive procedure that solutions of the Navier-Stokes exist locally in which two of the Cartesian components of velocity are essentially arbitrary. These solutions can be matched on a spherical surface to a well defined solution outside the surface of the flow equations such that all the physical quantities are continuous on the interface.

Introduction

The Navier-Stokes equations were first formulated by Navier [1] and some twenty years later by Stokes [2] using fewer assumptions with regard to the molecular interaction of fluids. Many authors have discussed exact solutions of these governing equations and reviews of this work have been given by Berker [3] and Wang [4], [5]. In particular a class of solutions considered by Weinbaum and O'Brien [6] in which the convective acceleration term is irrotational can be successfully employed for the solution of boundary value problems in the presence of a fixed sphere where the forcing flow is produced by an isolated flow singularity [7].

The motivation for the present discussion is to show that there are solutions which are locally valid and where two of the Cartesian components of velocity are essentially arbitrary, subject to preferential requirements of being locally continuous with their derivatives up to a finite order. The analysis which leads to this result is presented in three steps, and the fourth and final step is to construct the third velocity component uniquely by the application of integrability conditions. The construction of these solutions is tied closely to the standard theory of systems of linear-inhomogeneous partial differential equations [8]. It is found that these solutions are subject to basically two restrictions where two expressions are required to be non-vanishing in the fluid region. One such restriction excludes regions where there is a stagnation point, a point of zero vorticity, and motions where the velocity is orthogonal to the vorticity. The second restriction which is derived at a later point ensures that the third component of velocity is finite in the liquid.

In the last section of the paper it is shown how a general Beltrami force-free field type solution of the Navier-Stokes equations defined for the exterior region to a fixed spherical surface can be matched onto a random velocity field inside the sphere, such that the velocity, vorticity, normal and tangented stresses are continuous at the interface. This is possible because two of the velocity components are arbitrary and do not satisfy prescribed partial differential equations. Also the third component of velocity is uniquely determined in terms of the arbitrary components and their derivations through the application of integrability conditions. It is then possible to construct a random type velocity field locally which matches onto an otherwise well defined global solution of the Navier-Stokes equations outside the sphere.

[1] The dynamical system of equations representing the motion of a viscous incompressible liquid can be expressed by

$$\frac{\partial \underline{q}}{\partial t} + (\underline{q} \cdot \underline{\nabla})\underline{q} = -\nabla P + \nu \nabla^2 \underline{q} \qquad (1)$$

$$\operatorname{div} \underline{q} = 0 \quad ; \quad P = p/\rho_0 \; ; \qquad (2)$$

where $\underline{q} = u_j \hat{x}_j$ is the fluid velocity, p the pressure, ρ_0 the constant density, and ν the kinematic viscosity. It is also useful to define the Bernoulli function or total head of pressure by $B = P + \frac{1}{2}|\underline{q}|^2$.

The analysis can be started by considering the situation in which $u_1; u_2$ are arbitrarily prescribed and $u_3; P$ satisfy the linear inhomogeneous system

$$(\underline{q} \cdot \underline{\nabla})u_1 + P_{x_1} + \frac{\partial u_1}{\partial t} - \nu \nabla^2 u_1 = 0 \qquad (3)$$

$$\operatorname{div} \underline{q} = 0 \qquad (4)$$

Solutions of these equations exist and readily constructed by standard technique.

Now if $\underline{\gamma} = \underline{\omega} + \lambda\underline{q}$; $\underline{\omega} = \mathrm{curl}\,\underline{q}$; and λ is an arbitrary scalar function of position and time then

$$\begin{aligned}
\mathrm{div}\left\{\frac{\partial \underline{q}}{\partial t} + (\underline{q}\cdot\underline{\nabla})\underline{q} + \underline{\nabla}P - \nu\nabla^2\underline{q}\right\}
&= \nabla^2 B + \mathrm{div}[\underline{\omega}\times\underline{q}] = \nabla^2 B + \mathrm{div}[\underline{\gamma}\times\underline{q}] \\
&= \nabla^2 B + (\underline{q}\cdot\mathrm{curl}\,\underline{\gamma}) - (\underline{\gamma}\cdot\underline{\omega}) \\
&= \nabla^2 B + (\underline{q}\cdot\mathrm{curl}\,\underline{\gamma}) - |\underline{\omega}|^2 - \lambda(\underline{q}\cdot\underline{\omega}) : \qquad (5)
\end{aligned}$$

This expression vanishes if

$$\lambda = \frac{1}{(\underline{q}\cdot\underline{\omega})}\{\nabla^2 B + (\underline{q}\cdot\mathrm{curl}\,\underline{\gamma}) - |\underline{\omega}|^2\}; \qquad (6)$$

where $(\underline{q}\cdot\underline{\omega}) \neq 0$ in the fluid region. In this case it follows that

$$\underline{\gamma} = \underline{\omega} + \frac{\underline{q}}{\underline{q}\cdot\underline{\omega})}\{\nabla^2 B + (\underline{q}\cdot\mathrm{curl}\,\underline{\gamma}) - |\underline{\omega}|^2\}: \qquad (7)$$

To show that the equation for $\underline{\gamma}$ can be satisfied first set $\underline{\gamma} = \underline{\beta} + \underline{\nabla}\chi$, then (7) can be written as

$$\begin{aligned}
\underline{\beta} + \underline{\nabla}\chi = \underline{\omega} + {}& \qquad (8) \\
&\frac{\underline{q}}{(\underline{q}\cdot\underline{\omega})}\{\nabla^2 B + (\underline{q}\cdot\mathrm{curl}\,\underline{\beta}) - |\underline{\omega}|^2\}
\end{aligned}$$

and elimination of χ produces the equation

$$\begin{aligned}
\mathrm{curl}\,\underline{\beta} = {}& \mathrm{curl}\,\underline{\omega} \qquad (9) \\
&+ \mathrm{curl}\left\{\frac{\underline{q}}{(\underline{\omega}\cdot\underline{q})}[\nabla^2 B + (\underline{q}\cdot\mathrm{curl}\,\underline{\beta}) - |\underline{\omega}|^2]\right\} :
\end{aligned}$$

To show that equation (9) can be satisfied it is first convenient to set

$$\begin{aligned}
\underline{R} = R_j\hat{x}_j = {}& \underline{\beta} - \underline{\omega} - \frac{\underline{q}}{(\underline{\omega}\cdot\underline{q})} \qquad (10) \\
&\left(\nabla^2\beta + (\underline{q}\cdot\mathrm{curl}\,\underline{\beta}) - |\underline{\omega}|^2\right) - \underline{\nabla}G\chi
\end{aligned}$$

then the equations

$$\frac{\partial R_3}{\partial x_2} - \frac{\partial R_2}{\partial x_3} = (\hat{x}_1 \cdot \mathrm{curl}\,\underline{\beta}) + \left(\hat{x}_1 \cdot \mathrm{curl} \left\{ \frac{\underline{q}}{(\underline{\omega} \cdot \underline{q})} [\nabla^2 \underline{\beta} + (\underline{q} \cdot \mathrm{curl}\,\underline{\beta}) - |\underline{\omega}|^2] \right\} \right) = 0 ; \tag{11}$$

$$\frac{\partial R_1}{\partial x_3} - \frac{\partial R_3}{\partial x_1} = (\hat{x}_2 \cdot \mathrm{curl}\,\underline{\beta}) + \left(\hat{x}_2 \cdot \mathrm{curl} \left\{ \frac{\underline{q}}{(\underline{\omega} \cdot \underline{q})} [\nabla^2 \underline{\beta} + (\underline{q} \cdot \mathrm{curl}\,\underline{\beta}) - |\underline{\omega}|^2] \right\} \right) = 0 ; \tag{12}$$

are linear inhomogeneous and solutions exist for $\mathrm{curl}\,\underline{\beta}$ without restricting $\underline{q}, \underline{\beta}$ apart from $(\underline{\omega} \cdot \underline{q}) \neq 0$, in the fluid. In fact if $\underline{\zeta} = \zeta_j \hat{x}_j = \mathrm{curl}\,\underline{\beta}$, the system given by (11) and (12) can be recast as a first order system of the form

$$A_{ijk} \frac{\partial \zeta_i}{\partial x_j} + B_{ik} \zeta_i + C_k = 0 ; \tag{13}$$

for $k = 1, 2$ together with $\frac{\partial \zeta_i}{\partial x_i} = 0$, and the coefficients A_{ijk}, B_{ik}, C_k depend on u_j and B. Even though from a constructive approach it is a cumbersome procedure to exhibit the solutions for ζ_i explicitly it is sufficient for the present purpose to be assured that such solutions exist, and this is confirmed from standard theory (see [8]).

It now follows from (11)(12) that $\frac{\partial R_3}{\partial x_2} - \frac{\partial R_2}{\partial x_3} = 0$; $\frac{\partial R_1}{\partial x_3} - \frac{\partial R_3}{\partial x_1} = 0$, and it is always possible to choose R_j such that $\frac{\partial R_2}{\partial x_1} - \frac{\partial R_1}{\partial x_2} = 0$, in which case

$$(\hat{x}_3 \cdot \mathrm{curl}\,\underline{\beta}) + (\hat{x}_3 \cdot \mathrm{curl} \left\{ \frac{\underline{q}}{(\underline{\omega} \cdot \underline{q})} [\nabla^2 B + (\underline{q} \cdot \mathrm{curl}\,\underline{\beta}) - |\underline{\omega}|^2] \right\}) \tag{14}$$

and (9) is satisfied subject $(\underline{q} \cdot \underline{\omega}) \neq 0$ in the fluid. The meaning of this result is that the solution space of

$$\mathrm{div} \left\{ \frac{\partial \underline{q}}{\partial t} + (\underline{q} \cdot \underline{\nabla})\underline{q} + \underline{\nabla} P - \nu \nabla^2 \underline{q} \right\} = 0 ; \tag{15}$$

is sufficiently large as to encompass or include solutions of (3) and (4). In other words there is a mutually consistent solution of the equations (3) (4) (15), which will be identified together with supplementary conditions at a later point in the analysis.

[2] Consider now the linear-inhomogeneous system

$$\frac{\partial u_2}{\partial t} + (\underline{q}' \cdot \underline{\nabla})u_2 + \frac{\partial P'}{\partial x_2} - \nu \nabla^2 u_2 = 0\,; \quad \mathrm{div}\,\underline{q}' = 0\,; \tag{16}$$

where $\underline{q}' = u_1 \hat{x}_1 + u_2 \hat{x}_2 + u_3' \hat{x}_3$. The system contains u_3', P' as dependent variables and solutions can be constructed using straightforward methods. With $\underline{\gamma}' = \underline{\omega}' + \lambda' \underline{q}'$; $\underline{\omega}' = \mathrm{curl}\,\underline{q}'$. The preceding argument can be utilized to show that

$$\mathrm{div}\left\{\frac{\partial \underline{q}'}{\partial t} + (\underline{q}' \cdot \underline{\nabla})\underline{q}' + \underline{\nabla} P' - \nu \nabla^2 \underline{q}'\right\} = 0\,; \tag{17}$$

provided

$$\lambda' = \frac{1}{(\underline{\omega}' \cdot \underline{q}')}\left[\nabla^2 B' + (\underline{q}' \cdot \mathrm{curl}\,\underline{\gamma}') - |\underline{\omega}'|^2\right] : \tag{18}$$

In this case

$$\underline{\gamma}' = \underline{\omega}' + \frac{\underline{q}'}{(\underline{\omega}' \cdot \underline{q}')}\left[\nabla^2 B' + (\underline{q}' \cdot \mathrm{curl}\,\underline{\gamma}') - |\underline{\omega}'|^2\right]; \tag{19}$$

where $B' = P' + \frac{1}{2}|\underline{q}'|^2$. Again it is appropriate to set $\underline{\gamma}' = \underline{\beta}' + \underline{\nabla}\chi'$, so that

$$\underline{\beta}' + \underline{\nabla}\chi' = \underline{\omega}' + \frac{\underline{q}'}{(\underline{\omega}' \cdot \underline{q}')}\left[\nabla^2 B' + (\underline{q}' \cdot \mathrm{curl}\,\underline{\beta}') - |\underline{\omega}'|^2\right] \tag{20}$$

and elimination of χ' produces the equation

$$\begin{aligned}\mathrm{curl}\,\underline{\beta}' = {} & \mathrm{curl}\,\underline{\omega}' + \\ & \mathrm{curl}\left\{\frac{\underline{q}'}{(\underline{\omega}' \cdot \underline{q}')}\left[\nabla^2 B' + (\underline{q}' \cdot \mathrm{curl}\,\underline{\beta}') - |\underline{\omega}'|^2\right]\right\} :\end{aligned} \tag{21}$$

The previous argument can also be invoked to show that if

$$\underline{R}' = R_j'\hat{x}_j = \underline{\beta}' + \underline{\nabla} G_+' - \underline{\omega}' - \frac{\underline{q}'}{(\underline{\omega}'\cdot\underline{q}')}\left[\nabla^2 B' + (\underline{q}'\cdot\mathrm{curl}\,\underline{\beta}') - |\underline{\omega}'|^2\right] \tag{22}$$

then consider the equations

$$\left(\frac{\partial R_3'}{\partial x_2} - \frac{\partial R_2'}{\partial x_3}\right) = (\hat{x}_1\cdot\mathrm{curl}\,\underline{\beta}') + \left(\hat{x}_1\cdot\mathrm{curl}\left\{\frac{\underline{q}'}{(\underline{\omega}'\cdot\underline{q}')}\left[\nabla^2 B' + (\underline{q}'\cdot\mathrm{curl}\,\underline{\beta}') - |\underline{\omega}'|^2\right]\right\}\right) = 0. \tag{23}$$

$$\left(\frac{\partial R_1'}{\partial x_3} - \frac{\partial R_3'}{\partial x_1}\right) = (\hat{x}_2\cdot\mathrm{curl}\,\underline{\beta}') + \left(\hat{x}_2\cdot\mathrm{curl}\left\{\frac{\underline{q}'}{(\underline{\omega}'\cdot\underline{q}')}\left[\nabla^2 B' + (\underline{q}'\cdot\mathrm{curl}\,\underline{\beta}') - |\underline{\omega}'|^2\right]\right\}\right) = 0. \tag{24}$$

With $\underline{\zeta}' = \zeta_j'\hat{x}_j = \mathrm{curl}\,\underline{\beta}'$, the system represented by (23)(24) can be converted to a first order linear-inhomogeneous system of equations expressible in the form

$$A'_{ijk}\frac{\partial \zeta_j'}{\partial x_i} + B'_{jk}\zeta_j' + c_k' = 0; \quad k = 1, 2. \tag{25}$$

The local existence and construction of the functions basically depends only on the condition $(\underline{\omega}', \underline{q}') \neq 0$ in the liquid region since it can be assumed the coefficients in (25) are analytic in the liquid.

Since $\frac{\partial R_3'}{\partial x_2} - \frac{\partial R_2'}{\partial x_3} = 0$; $\frac{\partial R_1'}{\partial x_3} - \frac{\partial R_3'}{\partial x_1} \neq 0$, it follows that R_j' can always be chosen so that $\frac{\partial R_2'}{\partial x_1} - \frac{\partial R_1'}{\partial x_2} = 0$, in which case the net result is that there are mutually consistent solutions of the system

$$\mathrm{div}\left(\frac{\partial \underline{q}'}{dt} + (\underline{q}'\cdot\underline{\nabla})\underline{q}' + \underline{\nabla} P' - \nu\nabla^2\underline{q}'\right) = 0; \tag{26}$$

$$\frac{\partial u_2}{\partial t} + (\underline{q}' \cdot \underline{\nabla})u_2 + \frac{\partial P'}{\partial x_2} - \nu \nabla^2 u_2 = 0; \quad \operatorname{div} \underline{q}' = 0; \tag{27}$$

provided that $(\underline{\omega}' \cdot \underline{q}') \neq 0$ in the liquid.

[3] The next step in the analysis is to show that $u_3' = u_3$; $P' = P +$ constant. In order to achieve this result it is convenient to set $u_3' = u_3 + u$, so that $\underline{q}' = \underline{q} + u\hat{x}_3$. Now equations (3)(15)(16)(26) imply that

$$\frac{\partial \underline{q}}{\partial t} + (\underline{q} \cdot \underline{\nabla})\underline{q} + \underline{\nabla} P - \nu \nabla^2 \underline{q} = \operatorname{curl} \psi_1 \hat{x}_1 \tag{28}$$

$$\frac{\partial \underline{q}'}{\partial t} + (\underline{q}' \cdot \underline{\nabla})\underline{q}' + \underline{\nabla} P' - \nu \nabla^2 \underline{q}' = \operatorname{curl} \psi_2 \hat{x}_2 \tag{29}$$

$$\operatorname{div} \underline{q} = 0; \qquad \operatorname{div} \underline{q}' = 0. \tag{30}$$

By subtraction these equations imply that

$$\begin{aligned} u\frac{\partial \underline{q}}{\partial x_3} + u\frac{\partial u}{\partial x_3}\hat{x}_3 + (\underline{q} \cdot \underline{\nabla})u\hat{x}_3 \\ + \underline{\nabla}(P' - P) - \nu \nabla^2 u\hat{x}_3 = \operatorname{curl} \underline{S} \end{aligned} \tag{31}$$

$$\frac{\partial u}{\partial x_3} = 0; \quad \underline{S} = \psi_2 \hat{x}_2 - \psi_1 \hat{x}_1 + \underline{\nabla} \phi' = S_j \hat{x}_j \tag{32}$$

where the scalar functions $\psi_1; \psi_2; \phi'$ are arbitrary functions of position and time. If $\underline{S}$ is eliminated from (31) then

$$\operatorname{div} \left\{ u\frac{\partial \underline{q}}{\partial x_3} + u\frac{\partial u}{\partial x_3}\hat{x}_3 + (\underline{q} \cdot \underline{\nabla})u\hat{x}_3 \right\} + \nabla^2(P' - P) = 0. \tag{33}$$

Also by elimination of ψ_2 from [29] it is found that

$$\begin{aligned} &\left(\nu \nabla^2 - \frac{\partial}{\partial t}\right) \frac{\partial u_2}{\partial x_2} + \frac{\partial}{\partial x_1}(\underline{q} \cdot \underline{\nabla})u_1 + \frac{\partial}{\partial x_3}(\underline{q} \cdot \underline{\nabla})u_3 \\ &+ \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_3^2}\right) P' + \frac{\partial}{\partial x_1}\left(u\frac{\partial u_1}{\partial x_3}\right) + \frac{\partial}{\partial x_3}\left(u\frac{\partial u_2}{\partial x_3}\right) = 0. \end{aligned} \tag{34}$$

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)u_2 + (\underline{q}\cdot\nabla)u_2 + \frac{\partial P}{\partial x_2} + \left[\frac{\partial}{\partial x_2}(P' - P) + u\frac{\partial u_z}{\partial x_3}\right] = 0. \tag{35}$$

These equations imply

$$\begin{aligned}
&\left[\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)u_2 + (\underline{q}\cdot\underline{\nabla})u_2 + \frac{\partial P}{\partial x_2}\right]\cdot \\
&\left[\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_3^2}\right)P'' + \frac{\partial}{\partial x_1}\left(u\frac{\partial u_1}{\partial x_3}\right) + \frac{\partial}{\partial x_3}\left(u\frac{\partial u_3}{\partial x_3}\right)\right] - \\
&\left[\left(\nu\nabla^2 - \frac{\partial}{\partial t}\right)\frac{\partial u_2}{\partial x_2} + \frac{\partial}{\partial x_1}(\underline{q}\cdot\underline{\nabla})u_1 + \right. \\
&\left.\frac{\partial}{\partial x_3}(\underline{q}\cdot\underline{\nabla})u_3 + \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_3^2}\right)P\right]\left[\frac{\partial P''}{\partial x_2} + u\frac{\partial u_2}{\partial x_3}\right] = 0
\end{aligned} \tag{36}$$

where

$$P'' = P' - P \tag{37}$$

and the equation of continuity has been used to show that

$$\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_3}{\partial x_3}\right) = -\frac{\partial u_2}{\partial x_2}. \tag{38}$$

Regardless of whether one of the factors in (36) vanishes it follows that by the application of integrability conditions to this linear-homogeneous system the only consistent solution of (32)(33)(36) is that $u = 0$; or $u_3^0 = u_3$, and $P' = P +$ constant. The result is then expressed by the equation

$$\frac{\partial \underline{q}}{\partial t} + (\underline{q}\cdot\underline{\nabla})\underline{q} + \underline{\nabla}P - \nu\nabla^2\underline{q} = \mathrm{curl}\,\underline{T};\ \mathrm{div}\,\underline{q} = 0 \tag{39}$$

where $\underline{T} = \frac{1}{2}(\psi_1\hat{x}_1 + \psi_2\hat{x}_2) + \underline{\nabla}\phi''$, and

$$\frac{\partial T_3}{\partial x_2} - \frac{\partial T_2}{\partial x_3} = 0; \qquad \frac{\partial T_1}{\partial x_3} - \frac{\partial T_3}{\partial x_1} = 0. \tag{40}$$

Since $T_j;\ j = 1;2;3;$ can always be chosen so that $\frac{\partial T_2}{\partial x_1} - \frac{\partial T_1}{\partial x_2} = 0$, it follows that there exists a mutually consistent solution of the Navier-Stokes equations

$$\frac{\partial \underline{q}}{\partial t} + (\underline{q}\cdot\underline{\nabla})\underline{q} = -\underline{\nabla}P + \nu\nabla^2\underline{q};\ \mathrm{div}\,\underline{q} = 0; \tag{41}$$

in which $u_1;u_2$ are arbitrarily prescribed provided that $(\underline{\omega}\cdot\underline{q}) \neq 0$, in the liquid.

[4] It remains to construct the unknown velocity component by the application of integrability conditions. To this end it is appropriate to eliminate the pressure field P and consider the x_3-component of the vorticity equation. This may be written as

$$\begin{aligned}(\underline{q}\cdot\underline{\nabla})\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) - (\underline{\omega}\cdot\underline{\nabla})u_3 = \\ \left(\nu\nabla^2 - \frac{\partial}{\partial t}\right)\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right);\end{aligned} \tag{42}$$

where $\underline{\omega} = [\underline{\nabla}\times\underline{q}]$. This equation for the present purpose is more conveniently expressed

$$au_3 + b\frac{\partial u_3}{\partial x_1} + c\frac{\partial u_3}{\partial x_2} = d \tag{43}$$

where $a;b;c;d$ can be written in terms of $u_j;\ j = 1;2;$ derivatives by

$$a = \frac{\partial}{\partial x_3}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right);\quad b = \frac{\partial u_2}{\partial x_3};\quad c = -\frac{\partial u_1}{\partial x_3} \tag{44}$$

$$\begin{aligned}d = \left(\nu\nabla^2 - \frac{\partial}{\partial t} - u_1\frac{\partial}{\partial x_1} - u_2\frac{\partial}{\partial x_2}\right)\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) - \\ \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_2}\right).\end{aligned} \tag{45}$$

The functions $a;b;c;d;$ are all known in terms of $u_j;\ j = 1;2;$ and their derivatives. It is also observed that it is not necessary to consider the x_1 and x_2 – components of the vorticity equation since their satisfaction is guaranteed by the preceding analysis, and also they contain the higher order derivatives e.g.

$\frac{\partial^3 u_3}{\partial x_1 \partial x_2^2}; \frac{\partial^3 u_3}{\partial x_1^2 \partial x_2}$, which are not explicitly required by the integrability conditions. If equation (43) is differentiated with respect to x_3 then

$$a' u_3 + b' \frac{\partial u_3}{\partial x_1} + c' \frac{\partial u_3}{\partial x_2} = d'; \tag{46}$$

where

$$a' = \frac{\partial^2}{\partial x_3^2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right); \quad b' = \frac{\partial^2 u_2}{\partial x_3^2}; \quad c' = \frac{\partial^2 u_1}{\partial x_3^2}; \tag{47}$$

$$\begin{aligned} d' = \frac{\partial}{\partial x_3} & \left(\nu \nabla^2 - \frac{\partial}{\partial t} - u_1 \frac{\partial}{\partial x_1} - u_2 \frac{\partial}{\partial x_2}\right)\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) \\ & - \frac{\partial}{\partial x_3}\left\{\left(\frac{\frac{\partial u_2}{\partial x_1} - \partial u_1}{\partial x_2}\right)\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right)\right\} \\ & + \left(\frac{\partial u_2}{\partial x_3} \cdot \frac{\partial}{\partial x_1} - \frac{\partial u_1}{\partial x_3} \cdot \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)\right) \\ & \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right) : \end{aligned} \tag{48}$$

Once again the scalar functions a', b', c', d' are all known in terms of u_j; $j = 1, 2$; and their derivatives. One further differentiation of (46) is required with respect to x_3 and this can be represented by

$$a'' u_3 + b'' \frac{\partial u_3}{\partial x_1} + c'' \frac{\partial u_3}{\partial x_2} = d'' \tag{49}$$

where

$$a'' = \frac{\partial a'}{\partial x_3}; \quad b'' = \frac{\partial b'}{\partial x_1} + c'' = \frac{\partial c'}{\partial x_3} \tag{50}$$

$$d'' = \frac{\partial d'}{\partial x_3} - a' \frac{\partial u_3}{\partial x_3} - b' \frac{\partial^2 u_3}{\partial x_1 \partial x_3} - c' \frac{\partial^2 u_3}{\partial x_2 \partial x_3} : \tag{51}$$

In terms of u_1, u_2 these functions may be written as

$$a'' = \frac{\partial^3}{\partial x_3^3}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right); \quad b'' = \frac{\partial^3 u_2}{\partial x_3^3}; \quad c'' = -\frac{\partial^3 u_1}{\partial x_3^3}; \tag{52}$$

$$
\begin{aligned}
d'' = {} & \frac{\partial^2}{\partial x_3^2}\left\{\left(\nu\nabla^2 - \frac{\partial}{\partial t} - u_1\frac{\partial}{\partial x_1} - u_2\frac{\partial}{\partial x_2}\right)\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)\right\} \\
& + \frac{\partial}{\partial x_3}\left\{\frac{\partial u_2}{\partial x_3}\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right)\right\} - \\
& \frac{\partial}{\partial x_3}\left\{\frac{\partial u_1}{\partial x_3}\frac{\partial}{\partial x_2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right)\right\} \\
& + \frac{\partial}{\partial x_3}\left\{\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right)\frac{\partial}{\partial x_3}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)\right\} + \\
& \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right)\frac{\partial^2}{\partial x_3^2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) \\
& + \frac{\partial^2 u_2}{\partial x_3^2}\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right) - \frac{\partial^2 u_1}{\partial x_3^2}\frac{\partial}{\partial x_2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right) \\
& \frac{\partial^2}{\partial x_3^2}\left\{\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right)\right\}.
\end{aligned}
\tag{53}
$$

The solution for the velocity component u_3 is determined uniquely from equations (43)(46)(49) and is represented by

$$
u_3 = \frac{(dc' - cd')(bc'' - cb'') - (dc'' - cd'')(bc' - cb')}{(ac' - ca')(bc'' - cb'') - (ac'' - ca'')(bc' - cb')} \tag{54}
$$

and in order to be meaningful is subject to the additional restriction that the denominator of (54) is nonvanishing in the liquid or

$$
bc(c'a'' - a'c'') + ac(b'c'' - c'b'') + c^2(a'b'' - ba'') \neq 0. \tag{55}
$$

No major simplification arises from writing this expression in terms of u_1, u_2. In addition to (55) there is also the condition $(\underline{\omega} \cdot \underline{q}) \neq 0$ in the liquid. It is observed that this latter condition excludes two-dimensional motion where there exists a stream function and the vorticity is orthogonal to the plane of motion. In a similar manner axisymmetric flow is also excluded for the same reason. Clearly excluded are also flows where there is a stagnation point or a point where the vorticity vanishes. However it is possible to satisfy the above conditions with an axisymmetric flow containing swirl. In this case the velocity field $\underline{q}$ is of the form

$$
\underline{q} = \mathrm{curl}\,\frac{\psi_1}{\rho}\hat{\phi} + \frac{\psi_2}{\rho}\hat{\phi}; \tag{56}
$$

in cylindrical polar coordinates (z,ρ,ϕ), and ψ_j are both functions of (z,ρ,t). An example of such a flow has been given in [6] and $\mathrm{curl}\,\underline{q} = \alpha\underline{q}$ where α is a constant so that the convective acceleration term is irrotational. Specifically

$$\psi_1 = U(z,\rho)e^{-\alpha^2\nu t}; \quad \psi_2 = \alpha U(z,\rho)e_3^{-\alpha^2\nu t} \tag{57}$$

where U satisfies a type of reduced wave equation expressed by

$$(L_{-1} + \alpha^2) = 0; \quad L_{-1} \equiv \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial}{\partial \rho}. \tag{58}$$

In spherical polar coordinates $z = r\cos\theta$; $\rho = r\sin\theta$, a general solution is expressible in the form

$$U = r^{\frac{1}{2}} \sum_{n=1}^{\infty} A_n J_{n+\frac{1}{2}}(\alpha r)[P_{n-1}(\cos\theta) - P_{n+1}(\cos\theta)] \tag{59}$$

where $J_{n+\frac{1}{2}}(s)$ is the Bessel function of fractional order and $P_n(\cos\theta)$ is the Legendre polynomial. More generally there are three-dimensional asymmetric velocity fields which satisfy $\mathrm{curl}\,\underline{q} = \alpha\underline{q}$ and the solutions which satisfy the Navier-Stokes equations can be written as

$$\underline{q} = e^{-\alpha^2\nu t}\{\mathrm{curl}^2(A\underline{r}) + \alpha\,\mathrm{curl}\,(A\underline{r})\}; \tag{60}$$

where A satisfies

$$(\nabla^2 + \alpha^2)A = 0; \quad \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{61}$$

A separable solution in (r,θ,ϕ) coordinates is given by

$$A = r^{-\frac{1}{2}} J_{n+\frac{1}{2}}(\alpha r) P_n^m(\cos\theta) e^{im\phi} \tag{62}$$

where $P_n^m(\cos\theta)$ is the associated Legendre function.

The final point of interest in this presentation is that it is possible to match or patch up a solution of the type given by (62) exterior to a sphere $r = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}} = a$, with a solution for the velocity field described in sections [1] to [4] where u_1, u_2 are essentially arbitrary and do not satisfy partial differential equations. First it is possible to math up $u_1; u_2$ and all their derivatives to the order of the governing equations at $r = a$, with the corresponding values of $u_1; u_2$ and their derivatives from (62). Also since the derivation through equations (42)-(55) leads to a unique solution for u_3 because it satisfies the Navier-Stokes equations, it follows that u_3 and its derivatives up to the order of the flow equations can be matched at $r = a$. The velocity field, vorticity, normal and tangented stresses are then continuous at $r = a$. It is then possible to match a well defined solution outside a sphere with a solution of the flow equations inside a sphere where apart from the interface conditions at $r = a$, and the restrictions imposed by $(\underline{q} \cdot \underline{\omega}) \neq 0$ and (55) the motion is essentially arbitrary or random because of the arbitrary nature of $u_1; u_2$.

References

[1] Navier CLMH, Memoir on the laws of Motion of fluids, Mém. Acad. Sci. Paris 6 (1823): 389.

[2] Stokes, G G., On the theories of the internal friction of fluids in motion, Trans. Cambridge Phil. Soc. 8 (1845) 287-319.

[3] Berker, R., In Handbuch der Physick (Springer, Berline 1963) p 1-386.

[4] Wang, CY., Ann. Rev. Fluid Mech. 23 (1991) p. 159-177.

[5] ________, Appl. Mech. Rev. 42 (1989) p. 269-282.

[6] Weinbaum and V. O'Brien, Phys. Fluids 10 (1967) p. 1438-1447.

[7] Ranger, K B., Time-dependent decay of the motion of sphere translating and rotating in a viscous liquid, Quart Journal of Mechanical and Applied Math Vol. 49 Pt. 4 1996, p. 621-633.

[8] Courant, R. and Hibert, D., Methods of Mathematical Physics Vol. 2, Interscience Publishers 1962, p. 22-147.