

Helicity and the Electromagnetic Field

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The structure of the Poincaré group gives, under all conditions, an equation of field helicity which reduces to the Maxwell equations and also gives cyclic relations between field components. If the underlying symmetry of special relativity is represented by the Poincaré group, it follows that the Maxwell equations and the cyclic equations are both products of special relativity itself, and both stem from the equation of helicity. This means that the symmetry of special relativity demands the existence of longitudinal solutions of Maxwell's equations under all topological conditions. In particular, the fundamental spin component of the electromagnetic field is $\mathbf{B}^{(3)}$, a longitudinal magnetic flux density which is free of phase and which is a topological invariant.

Key Words: Helicity equation; Poincaré group; $\mathbf{B}^{(3)}$ field.

1. Introduction

The first principle on which this paper is based is that a theory be developed according to its fundamental underlying symmetry: for the electromagnetic field this is the symmetry of special relativity [1–5], a sub-symmetry of general relativity. We accept the Poincaré group as the group of special relativity, with ten generators and two invariants [6]. The electromagnetic field is considered to be a physical entity which is described by symmetry-guided relations between group generators according to the following prescription [1–5]. Rotation generators are those of magnetic field components; boost generators are those of electric field components; translation generators are those of four potential components. It is shown in Sec. 2 that the Lie algebra of the Poincaré group leads to relations between generator eigenvalues which, using the above prescription, are consistent with the Maxwell equations and recently inferred [1–5] cyclic relations between field components.

Section 3 develops a helicity equation [7] from the underlying symmetry of the Poincaré group as given in Sec. 2. This equation has been inferred independently by Dvoeglazov [8] and by Afanasev and Stepanofsky [9], following the introduction of relativistic field helicity by Ranada [10], and the earlier realization that helicity is a topological invariant [11]. The transition from the static symmetry characteristics of the Poincaré group to an equation of motion (the helicity equation) is accomplished through the transition from momentum to coordinate representation P^μ is replaced by $i\mathbb{1}^m$, where P^m is the translation generator. This is synonymous with the

well known quantum hypothesis, which is a successful calculating prescription in field theory and wave mechanics. This transition changes the fundamental group identity [12],

$$P_\mu \tilde{W}^\mu = 0 \quad (1)$$

to

$$i\mathbb{1}^m \tilde{w}^m = 0 \quad (2)$$

giving the structure of the helicity equation (2) directly from the operator identity (1) the orthogonality identity of the Poincaré group [12]. Here \tilde{W}^m is the Pauli-Lyubanski (PL) operator and \tilde{w}^m its eigenvalue. The operator \tilde{W}^m generates relativistic helicity, being the tensor product of rotation and boost generators with P^m . Using the prescription developed in Sec. 2 it generates the relativistic field helicity vector. The Maxwell equations and cyclic equations follow from this principle, which applies the complete known symmetry of special relativity [12] to the electromagnetic field.

Section 3 uses the principle to show that the helicity of the field in the vacuum (charge free region) is given by a PL vector whose only non-zero component is proportional to $\mathbf{B}^{(3)}$, and so the helicity of the field vanishes if $\mathbf{B}^{(3)}$ vanishes, as asserted in the received view of electrodynamics [13–15]. However, if the helicity vanishes, there remains no physical or topological entity, *i.e.*, there is no field present at all, a self-inconsistency. There exists therefore a topologically invariant $\mathbf{B}^{(3)}$ if there exists a topologically invariant helicity. Thus $\mathbf{B}^{(3)}$ is the phase free, invariant, spin field of vacuum electromagnetism.

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It is no longer possible to accept the received view (transverse components only) because this view leads to a self inconsistency which cannot be rectified without the introduction and recognition of $\mathbf{B}^{(3)}$ as the fundamental magnetic flux density. Transverse components exist because $\mathbf{B}^{(3)}$ exists, and $\mathbf{B}^{(3)}$ is the simplest representation of the electromagnetic spin. Analogously, one axis of the Cartesian frame exists because the other two exist, and so on in cyclic permutation: in the last analysis therefore the reason for the existence of $\mathbf{B}^{(3)}$ is as simple as this.

2. Symmetry, B Cyclics and Maxwell Equations

In order to develop the structural characteristics of the Poincaré group the opening part of this section is devoted to its Lie algebra [12], *i.e.*, to the commutative properties of P^m and \tilde{W}^m . It will be shown that the complete Lie algebra is,

$$[P_\mu, P_\nu] = 0, \quad (3)$$

$$[P_m, \tilde{W}_n] = 0,$$

for all m, n , and,

$$\begin{cases} [\tilde{W}_0, \tilde{W}_1] = -i(\tilde{W}_2 P_3 - \tilde{W}_3 P_2) = i(P_3 \tilde{W}_2 - P_2 \tilde{W}_3), \\ [\tilde{W}_0, \tilde{W}_2] = -i(\tilde{W}_3 P_1 - \tilde{W}_1 P_3) = i(P_1 \tilde{W}_3 - P_3 \tilde{W}_1), \\ [\tilde{W}_0, \tilde{W}_3] = -i(\tilde{W}_1 P_2 - \tilde{W}_2 P_1) = i(P_2 \tilde{W}_1 - P_1 \tilde{W}_2), \\ [\tilde{W}_1, \tilde{W}_2] = i(\tilde{W}_3 P_0 - \tilde{W}_0 P_3) = i(P_0 \tilde{W}_3 - P_3 \tilde{W}_0), \\ [\tilde{W}_2, \tilde{W}_3] = i(\tilde{W}_1 P_0 - \tilde{W}_0 P_1) = i(P_0 \tilde{W}_1 - P_1 \tilde{W}_0), \\ [\tilde{W}_3, \tilde{W}_1] = i(\tilde{W}_2 P_0 - \tilde{W}_0 P_2) = i(P_0 \tilde{W}_2 - P_2 \tilde{W}_0). \end{cases} \quad (5)$$

In order to derive Eqs. (3) to (5) we have used the commutator relations

$$[P_\mu, J_{\rho\sigma}] = i(g_{\mu\rho} P_\sigma - g_{\mu\sigma} P_\rho), \quad (6)$$

and

$$[\tilde{W}_\mu, J_{\rho\sigma}] = i(g_{\mu\sigma} \tilde{W}_\rho - g_{\mu\rho} \tilde{W}_\sigma), \quad (7)$$

where $g_{\mu\nu}$ is the metric tensor. In these relations the Pauli Lyubanski vector is defined by [12-15],

$$\tilde{W}_\mu := -\frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} J^{\nu\rho} P^\sigma. \quad (8)$$

Therefore the PL four-vector is made up of sums of quadratic products of operators (group generators). Its \tilde{W}_0 component is the scalar helicity operator in particle physics. The P and \tilde{W} vectors form the two Casimir invariants [12-15]

$$\begin{cases} \tilde{W}_0 = -J_1 P_1 - J_2 P_2 - J_3 P_3, \\ \tilde{W}_1 = J_1 P_0 + K_2 P_3 - K_3 P_2, \\ \tilde{W}_2 = J_2 P_0 + K_3 P_1 - K_1 P_3, \\ \tilde{W}_3 = J_3 P_0 + K_1 P_2 - K_2 P_1 \end{cases} \quad (9)$$

of the Poincaré group, the mass and spin invariant. All particles, including the photon, are classified in terms of these invariants.

In order to arrive at this Lie algebra we have used the rules governing the algebra of commutator:

$$\begin{aligned} [A, B] &= [B, A], \\ [A, BC] &= [A, B]C + B[A, C], \\ [AB, C] &= A[B, C] + [A, C]B, \\ [DE, BC] &= [D, B]EC + [E, B]C + B[D, C]E + BD[A, C], \end{aligned}$$

The algebra (3) and (4) shows that all components of P_μ commute with all components of \tilde{W}_ν in cyclic permutation. Within this seemingly simple structure occur, however, cyclic relations such as,

$$\begin{cases} [P_1, \tilde{W}_2] = i[P_3, P_0], \\ [P_2, \tilde{W}_3] = i[P_1, P_0], \\ [P_3, \tilde{W}_1] = i[P_2, P_0]. \end{cases} \quad (11)$$

There are also cyclic relations inherent in the sub-algebra (3) to (5), in which the rotation matrix is defined as a matrix of generators as follows,

$$\begin{aligned} J^{\mu\nu} &:= \begin{bmatrix} 0 & -K_1 & -K_2 & -K_3 \\ K_1 & 0 & -J_3 & J_2 \\ K_2 & J_3 & 0 & -J_1 \\ K_3 & -J_2 & J_1 & 0 \end{bmatrix}, \\ J_{\mu\nu} &:= \begin{bmatrix} 0 & K_1 & K_2 & K_3 \\ -K_1 & 0 & -J_3 & J_2 \\ -K_2 & J_3 & 0 & -J_1 \\ -K_3 & -J_2 & J_1 & 0 \end{bmatrix} \end{aligned} \quad (12)$$

giving the duals:

$$\tilde{J}^{\mu\nu} := \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} J_{\rho\sigma} = \begin{bmatrix} 0 & -J_1 & -J_2 & -J_3 \\ J_1 & 0 & K_3 & -K_2 \\ J_2 & -K_3 & 0 & K_1 \\ J_3 & K_2 & -K_1 & 0 \end{bmatrix}, \quad (13)$$

$$\tilde{J}_{\mu\nu} := \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} J^{\rho\sigma} = \begin{bmatrix} 0 & J_1 & J_2 & J_3 \\ -J_1 & 0 & K_3 & -K_2 \\ -J_2 & -K_3 & 0 & K_1 \\ -J_3 & K_2 & -K_1 & 0 \end{bmatrix}$$

In the lightlike condition

$$P^\mu = (P_0, 0, 0, P_0), \quad P_3 = P_0 \quad (14)$$

and

$$\begin{cases} \tilde{W}_0 = -J_3 P_3, \\ \tilde{W}_1 = J_1 P_0 + K_2 P_3, \\ \tilde{W}_2 = J_2 P_0 - K_1 P_3, \\ \tilde{W}_3 = J_3 P_0. \end{cases} \quad (15)$$

In this condition therefore,

$$[\tilde{W}_1, \tilde{W}_2] = P_0[J_1 + K_2, J_2 - K_1] = 0, \quad (16)$$

which is consistent with the Lie algebra of the Lorentz group

$$[J_1, J_2] + [K_1, K_2] + [K_2, J_2] + [K_1, J_1] = 0 \quad (17)$$

Equation (16) is also consistent with

$$J_1 + K_2 = J_2 - K_1 = 0. \quad (18)$$

The overall Lie algebra also contains an $E(2)$ structure, namely,

$$\begin{aligned} [\tilde{W}_1, \tilde{W}_2] &= i(\tilde{W}_3 P_0 + \tilde{W}_0 P_3), \\ [J_3, \tilde{W}_1] &= i\tilde{W}_2, \\ [\tilde{W}_2, J_3] &= i\tilde{W}_1, \end{aligned} \quad (19)$$

which in the lightlike condition becomes

$$\begin{aligned} [\tilde{W}_1, \tilde{W}_2] &= 0, & [J_3, \tilde{W}_1] &= i\tilde{W}_2 \\ [\tilde{W}_2, J_3] &= i\tilde{W}_1, \end{aligned} \quad (20)$$

the planar Euclidean group. The latter is the little group in the lightlike condition [12-15]. It is seen that J_3 is non-zero and in the field interpretation $\mathbf{B}^{(3)}$ is non-zero in the $E(2)$ group.

If there is a rest frame,

$$P^\mu = (P_0, 0, 0, 0), \quad (21)$$

and Eq. (5) becomes an $O(3)$ structure

$$\begin{aligned} [\tilde{W}_1, \tilde{W}_2] &= iP_0 \tilde{W}_3, & [\tilde{W}_2, \tilde{W}_3] &= iP_0 \tilde{W}_1 \\ [\tilde{W}_3, \tilde{W}_1] &= iP_0 \tilde{W}_2, \end{aligned} \quad (22)$$

In the rest frame, however,

$$\begin{aligned} W_0 &= 0, & W_1 &= J_1 P_0, \\ W_2 &= J_2 P_0, & W_3 &= J_3 P_0 \end{aligned} \quad (23)$$

and the $O(3)$ structure (22) becomes the cyclic Lie algebra of the rotation generators of the Lorentz group. Therefore a complete knowledge of the Lie algebra shows that the $E(2)$ and $O(3)$ groups can be generated as sub algebra of the Poincaré group's Lie algebra.

The complete PL vector in the lightlike condition is therefore

$$W^\mu = P_0(-J_3, 0, 0, J_3), \quad (24)$$

and if we accept the constraints,

$$J_1 = -K_2, \quad J_2 = K_1, \quad (25)$$

which in the field interpretation are given by the Faraday law of induction, *i.e.*

$$cB_1 = -E_2, \quad cB_2 = E_1. \quad (26)$$

This is a simple illustration of the fact that the experimentally verified Faraday law of induction leads to the conclusion that a non-zero $\mathbf{B}^{(3)}$ is needed for nonzero field helicity. Since helicity is a topological invariant, $\mathbf{B}^{(3)}$ is non-zero topologically.

If we accept the first principle that all theories in special relativity are based on the underlying Poincaré group, we could proceed logically by deriving the equations of

the electromagnetic field from the group structure, as just illustrated for the Faraday law. The Lie algebra includes that of $E(2)$ and $O(3)$, and applies to all physical entities and theories within special relativity, using vectors and spinors. The notion of the relativistic helicity of the classical electromagnetic field is based on the existence of P_μ

and \tilde{W}_μ , and leads to the existence of $\mathbf{B}^{(3)}$ as a topological invariant. The same group structure shows that $\mathbf{B}^{(3)*}$ must always be related to $\mathbf{B}^{(4)} = \mathbf{B}^{(2)*}$ topologically, and this determines the way in which $\mathbf{B}^{(3)}$ interacts with a fermion [16] as in the inverse Faraday effect. Dvoeglazov has recently developed several theories based on field and particle helicity and chirality [8]. Any generalization of the Maxwell equations must take place within the Poincaré group if we proceed within special relativity. In general relativity the underlying symmetry group becomes the Einstein group.

Part of the Lie algebra given above has the structure of the four Maxwell equations, and another part gives the structure of the cyclic relations between field components now known to be an intrinsic feature of electromagnetism [1-5]. For example, consider the commutators,

$$\begin{aligned} [P_2, J_3] &= iP_1, & [P_3, J_2] &= -iP_1 \\ [P_0, K_1] &= iP_1 \end{aligned} \quad (27)$$

Using the coordinate representation of the translation generator [12]:

$$P_\mu = i\partial_\mu, \quad (28)$$

Eq. (27) becomes

$$([\partial_2, J_3] - [\partial_3, J_2] - [\partial_0, K_1])\psi = P_1\psi, \quad (29)$$

where ψ is an eigenfunction. Equation (29) can be rewritten as

$$(\partial_2 J_3 - \partial_3 J_2 - \partial_0 K_1 - (J_3 \partial_3 - J_2 \partial_2 - K_1 \partial_0))\psi = P_1\psi, \quad (30)$$

which is a relation between operators on ψ . We use

$$J_3\psi = j_3\psi, \quad J_2\psi = j_2\psi, \quad K_1\psi = k_1\psi, \quad (31)$$

where lower case letters denote eigenvalues. We have

$$\begin{aligned} \partial_2(j_3\psi) &= (\partial_2 j_3)\psi + j_3(\partial_2\psi), \\ \partial_3(j_2\psi) &= (\partial_3 j_2)\psi + j_2(\partial_3\psi), \\ \partial_0(k_1\psi) &= (\partial_0 k_1)\psi + k_1(\partial_0\psi). \end{aligned} \quad (32)$$

It is now assumed that

$$\begin{aligned} J_3(\partial_2\psi) + J_2(\partial_3\psi) + K_1(\partial_0\psi) \\ = j_3(\partial_2\psi) + j_2(\partial_3\psi) + k_1(\partial_0\psi), \end{aligned} \quad (33)$$

which is compatible with

$$(\partial_2 + \partial_3 + \partial_0)\psi = \text{constant } \psi. \quad (34)$$

Equations (30) to (34) give the eigenvalue relation

$$\partial_2 j_3 - \partial_3 j_2 - \partial_0 k_1 = p_1, \quad (35)$$

which is one component of the vector equation

$$\nabla \times \mathbf{j} - \frac{1}{c} \frac{\partial \mathbf{k}}{\partial t} = \mathbf{p}. \quad (36)$$

This equation has the same structure exactly as the Ampère law extended with Maxwell's displacement current. The eigenvalue \mathbf{j} represents the magnetic field, \mathbf{k} the electric field, \mathbf{p} the current (or potential vector). If we write

$$\psi := e^{i\phi} \psi_0, \quad (37a)$$

where ϕ is a phase factor, then,

$$\hat{J}_3(e^{i\phi} \psi_0) = j_3^{(0)} e^{i\phi} \psi_0 = j_3 \psi, \quad (37b)$$

and so on. Therefore the eigenvalues appearing in Eq. (36) are phase dependent in general.

The complete set of operator relations leading to this equation is

$$\begin{aligned} ([\partial_1, J_2] - [\partial_2, J_1] - [\partial_0, K_3])\psi &= P_3\psi, \\ ([\partial_2, J_3] - [\partial_3, J_2] - [\partial_0, K_1])\psi &= P_1\psi, \\ ([\partial_3, J_1] - [\partial_1, J_3] - [\partial_0, K_2])\psi &= P_2\psi. \end{aligned} \quad (38)$$

Similarly, the Lie algebra

$$([\partial_2, K_3] - [\partial_3, K_2] + [\partial_0, J_3])\psi = 0, \quad (39)$$

and so forth leads to the following relation between eigenvalues of the group generators

$$\nabla \times \mathbf{k} + \frac{1}{c} \frac{\partial \mathbf{j}}{\partial t} = 0. \quad (40)$$

This equation has the same structure as the Faraday law as discussed already.

The Lie algebra,

$$([\partial_1, J_1] + [\partial_2, J_2] + [\partial_3, J_3])\psi = 0. \quad (41)$$

gives

$$\left(\begin{aligned} &(\partial_1 J_1 - J_1 \partial_1) + (\partial_2 J_2 - J_2 \partial_2) \\ &+ (\partial_3 J_3 - J_3 \partial_3) \end{aligned} \right) \psi = 0. \quad (42)$$

Using

$$\begin{aligned} J_1 \psi &= j_1 \psi, \\ \partial_1(j_1 \psi) &= j_1(\partial_1 \psi) + (\partial_1 j_1) \psi, \end{aligned} \quad (43)$$

and assuming that

$$\begin{aligned} &J_1(\partial_1 \psi) + J_2(\partial_2 \psi) + J_3(\partial_3 \psi) \\ &= j_1(\partial_1 \psi) + j_2(\partial_2 \psi) + j_3(\partial_3 \psi), \end{aligned} \quad (44)$$

leads to the structure of the third Maxwell equation,

$$\partial_1 j_1 + \partial_2 j_2 + \partial_3 j_3 = 0, \quad (45)$$

i.e. $\nabla \cdot \mathbf{j} = 0$

Finally,

$$([\partial_1, K_1] + [\partial_2, K_2] + [\partial_3, K_3])\psi = 3P_0\psi, \quad (46)$$

leads to

$$\nabla \cdot \mathbf{k} = 3p_0,$$

assuming that

$$\begin{aligned} &K_1(\partial_1 \psi) + K_2(\partial_2 \psi) + K_3(\partial_3 \psi) \\ &= k_1(\partial_1 \psi) + k_2(\partial_2 \psi) + k_3(\partial_3 \psi) \end{aligned} \quad (48)$$

Therefore all four Maxwell equations emerge from the Lie algebra of the Poincaré group, *i.e.*,

$$\begin{aligned} \nabla \cdot \mathbf{j} &= 0, \quad \nabla \times \mathbf{k} + \frac{1}{c} \frac{\partial \mathbf{j}}{\partial t} = 0, \\ \nabla \times \mathbf{j} + \frac{1}{c} \frac{\partial \mathbf{k}}{\partial t} &= \mathbf{p}, \quad \nabla \cdot \mathbf{k} = 3p_0. \end{aligned} \quad (49)$$

It is important to note that the complete Poincaré group (inclusive of the translation generator) is needed to obtain the complete structure of the Maxwell equations. In particular, the structure of the group allows the existence of the Lehnert current, which is the non-zero vacuum divergence of the electric field in Maxwell's equations [4]. This is seen in Eq. (49) through the term $3p_0$. The Lehnert current is therefore intrinsic within the structure of the Poincaré group but not that of the Lorentz group, in which there is no translation generator.

In particular, the boost operators take the place of electric field components and the rotation generators take the place of magnetic field components. This suggests that the generators act on eigenfunctions to give field components as eigenvalues. The P^μ generator is also an operator [12] and J_μ , K_μ and P_μ are special cases of [12],

$$X_\alpha := i \left(\begin{aligned} &\left. \frac{\partial x'}{\partial a^\alpha} \right|_{a=0} \frac{\partial}{\partial x} + \left. \frac{\partial y'}{\partial a^\alpha} \right|_{a=0} \frac{\partial}{\partial y} + \left. \frac{\partial z'}{\partial a^\alpha} \right|_{a=0} \frac{\partial}{\partial z} \\ &\left. \frac{\partial}{\partial y} + \frac{\partial z'}{\partial a^\alpha} \right|_{a=0} \frac{1}{c} \frac{\partial}{\partial t} \end{aligned} \right) \quad (50)$$

Equation (10) defines the generator corresponding to the parameter a^α of the r parameter group ($\alpha = 1, \dots, r$); and X_α within the Poincaré group must be consistent with the most general type of Lorentz transform

$$x'^\mu = \Lambda_\nu^\mu x^\nu + a^\mu, \quad (51)$$

where Λ_ν^μ includes boosts and rotations, and where a^μ describes space-time translations.

The operator products $P_\mu P^\mu$, $\tilde{W}_\mu \tilde{W}^\mu$, and $P_\mu \tilde{W}^\mu$ are invariant under Lorentz transformation [15,12]. The product $P_\mu \tilde{W}^\mu$ is always zero by definition (Eqs. (1) and (7)). It follows that the commutator relations (3) to (5) must also be Lorentz covariant, and if they are zero in one frame they are zero in all frames. The Lie algebra of the Poincaré group is Lorentz covariant by definition, because the Poincaré group is the group of special relativity itself. Since P_μ and W_μ are operators, the correct commutator algebra must be used, represented by relations such as Eq. (10).

The commutator of \tilde{W}_0 and \tilde{W}_1 is given for example by

$$\begin{aligned} [W_0, W_1] &= [W_1, J_1 P_1 + J_2 P_2 + J_3 P_3] \\ &= [W_1, J_1] P_1 + [W_1, J_2] P_2 + [W_1, J_3] P_3 \\ &= i(W_3 P_2 - W_2 P_3) \end{aligned} \quad (52)$$

3. Relativistic Helicity, B Cyclics and Maxwell Equations

The relativistic helicity of the classical electromagnetic field is defined through the \tilde{W}_μ vector in Eq. (8), which is the equation that essentially transforms the Lorentz group into the Poincaré group by adjoining the operator P_μ [12]. Therefore the relativistic helicity cannot be defined without consideration of space-time translation. In the field interpretation it cannot be defined in terms of the antisymmetric field tensor alone, and this is why $\mathbf{B}^{(3)}$ the field's fundamental nature does not manifest itself in the Lorentz group. It was well known that the P_μ operator was not introduced until 1939 [6], so the $\mathbf{B}^{(3)}$ field could not have been understood during the formative years of relativistic electrodynamics.

It is convenient to rewrite Eq. (8) using the dual defined in Eq. (13):

$$\text{a) } \tilde{W}_\mu := -\tilde{J}_{\mu\nu} P^\nu = - \begin{bmatrix} 0 & J_1 & J_2 & J_3 \\ -J_1 & 0 & K_3 & -K_2 \\ -J_2 & -K_3 & 0 & K_1 \\ -J_3 & K_2 & -K_1 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} \quad (53a)$$

$$\text{b) } \tilde{W}^\mu := -\tilde{J}^{\mu\nu} P_\nu = \begin{bmatrix} 0 & -J_1 & -J_2 & -J_3 \\ J_1 & 0 & K_3 & -K_2 \\ J_2 & -K_3 & 0 & K_1 \\ J_3 & K_2 & -K_1 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ -P_1 \\ -P_2 \\ -P_3 \end{bmatrix} \quad (53b)$$

The products $P_\mu P^\mu$, $\tilde{W}_\mu \tilde{W}^\mu$ and $P_\mu \tilde{W}^\mu$ are invariants of the Poincaré group, which suggests that for the classical electromagnetic field, there exists the helicity four-vector [7-11],

$$\tilde{G}^\mu := \tilde{G}^{\mu\nu} A_\nu \quad (54)$$

whose structure is analogous to that of Eq. (53b), *i.e.*,

$$\tilde{G}^\mu := \begin{bmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & E_3 & -E_2 \\ B_2 & -E_3 & 0 & E_1 \\ B_3 & E_2 & -E_1 & 0 \end{bmatrix} \begin{bmatrix} A_0 \\ -A_1 \\ -A_2 \\ -A_3 \end{bmatrix} = \begin{bmatrix} B_1 A_1 + B_2 A_2 + B_3 A_3 \\ B_1 A_0 - E_3 A_2 + E_2 A_3 \\ B_2 A_0 + E_3 A_1 - E_1 A_3 \\ B_3 A_0 - E_2 A_1 + E_1 A_2 \end{bmatrix} \quad (55)$$

It is clear that the \tilde{G}^μ vector in the field interpretation plays the role of the \tilde{W}^μ vector in the particle interpretation of the electromagnetic entity, considered to be a physical entity. From the field-particle dualism of \tilde{G}^μ and \tilde{W}^μ it is inferred that the quantities $A_\mu A^\mu$, $\tilde{G}_\mu \tilde{G}^\mu$,

and $A_\mu \tilde{G}^\mu$ are invariants of the Poincaré group. In particular,

$$P_\mu \tilde{W}^\mu = A_\mu \tilde{G}^\mu = 0, \quad (56)$$

which expresses the orthogonality between operators. Reinstating the unwritten eigenfunction,

$$P_\mu \tilde{W}^\mu \Psi = P_\mu (\tilde{\omega}^\mu \Psi) = 0, \quad (57)$$

where $\tilde{\omega}^\mu$ is the eigenvalue corresponding to the eigenoperator \tilde{W}^μ . Taking the coordinate representation of P_μ [12],

$$P_\mu := i\partial_\mu, \quad (58a)$$

means that

$$\partial_\mu (\tilde{\omega}^\mu \Psi) = (\partial_\mu \tilde{\omega}^\mu) \Psi + \tilde{\omega}^\mu \partial_\mu \Psi = 0. \quad (58b)$$

If we assume that

$$\partial_\mu \tilde{\omega}^\mu = 0, \quad (59)$$

a conservation equation is obtained for the eigenvalue of the operator \tilde{W}^μ . If it is assumed that the same conservation equation is true for \tilde{G}^μ , regarded as an eigenvalue, the Maxwell equations result. This is demonstrated as follows. The vector form of the equation

$$\partial_\mu \tilde{G}^\mu = 0, \quad (60a)$$

is, in S.I. units

$$\frac{1}{c} \frac{\partial}{\partial t} (\mathbf{A} \cdot \mathbf{B}) + \nabla \cdot (A_0 \mathbf{B}) + \frac{1}{c} \nabla \cdot (\mathbf{E} \times \mathbf{A}) = 0. \quad (60b)$$

Now use the vector identities,

$$\begin{aligned} \frac{\partial}{\partial t} (\mathbf{A} \cdot \mathbf{B}) &= \mathbf{A} \cdot \frac{\partial \mathbf{B}}{\partial t} + \mathbf{B} \cdot \frac{\partial \mathbf{A}}{\partial t}, \\ \nabla \cdot (A_0 \mathbf{B}) &= A_0 \nabla \cdot \mathbf{B} + \mathbf{B} \cdot \nabla A_0, \\ \nabla \cdot (\mathbf{E} \times \mathbf{A}) &= \mathbf{A} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{A}), \end{aligned} \quad (61)$$

to find that

$$\begin{aligned} \frac{1}{c} \mathbf{A} \cdot \left(\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} \right) + A_0 \nabla \cdot \mathbf{B} + \frac{1}{c} \mathbf{B} \cdot \frac{\partial \mathbf{A}}{\partial t} \\ - \frac{1}{c} \mathbf{E} \cdot (\nabla \times \mathbf{A}) + \mathbf{B} \cdot \nabla A_0 = 0. \end{aligned} \quad (62)$$

A particular solution of Eq. (62) is

$$\begin{aligned} \nabla \cdot \mathbf{B} = 0, \quad \frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0, \\ \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \nabla A_0, \quad \mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} \cdot \mathbf{B} = 0 \end{aligned} \quad (63)$$

which lists two of the Maxwell equations, defines the fields \mathbf{E} and \mathbf{B} in terms of A_μ , and uses the assumption $\mathbf{E} \perp \mathbf{B}$. Equation (63) is given in the received view as relations between transverse fields \mathbf{E} and \mathbf{B} , the fundamental components of the electromagnetic field under any condition. It is clear that Eqs. (63) are special solutions of Eq. (60a), thus justifying the latter empirically.

However, if the usual transverse solutions and transverse gauge [12] are used in the definition of \tilde{G}^μ , Eq. (55), we obtain,

$$\begin{aligned} A_0 = A_3 = E_3 = B_3 = 0, \quad B_1 = iB_2 = i\frac{B_0}{\sqrt{2}}e^{i\phi} \\ A_1 = iA_2 = i\frac{A_0}{\sqrt{2}}e^{i\phi}, \quad E_1 = iE_2 = \frac{E_0}{\sqrt{2}}e^{i\phi}, \end{aligned} \quad (64)$$

and

$$\tilde{G}^\mu = ?(0,0,0,0), \quad (65)$$

despite the fact that Eq. (64) is consistent with Eq. (63).

This is a fundamental paradox of the accepted theory of classical electrodynamics: the use of transverse plane waves and transverse gauge leads to the complete loss of the vector dual of the field, *i.e.*, \tilde{G}^μ is a null vector for transverse plane waves. This is inconsistent with the fact that is a topological invariant, and plainly inconsistent with the fact that the equations (63) were obtained from a non-zero field vector \tilde{G}^μ whose structure is as follows:

$$\tilde{G}^\mu = \begin{pmatrix} B_1A_1 + B_2A_2 + B_3A_3, B_1A_0 - E_3A_2 + E_2A_3, \\ B_2A_0 + E_3A_1 - E_1A_3, B_3A_0 - E_2A_1 + E_1A_2 \end{pmatrix} \quad (66)$$

This structure contains longitudinal (3) as well as transverse (1,2) components. Furthermore, the structure (66) is derived directly from the field representation of the Pauli Lyubanski vector in the Poincaré group. Without \tilde{W}^μ or \tilde{G}^μ the Poincaré group does not exist, and is replaced by the Lorentz group. The helicity vector \tilde{G}^μ originates therefore in the fact that the electromagnetic field both rotates and translates. These considerations suggest that Eq. (60a) provides solutions for fields that are missing in conventional electrodynamics. One of these is $\mathbf{B}^{(3)}$, another is the Coulomb field, both of which are absent from the particular solutions (64). From the definition of the Poincaré group

$$A_0\tilde{G}^\mu := \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}G^{\nu\rho}A^\sigma, \quad (67)$$

is a non-linear and cyclic relation, which for simplicity can be reduced to

$$\tilde{G}_\mu = \tilde{G}_{\mu\nu}\varepsilon^\nu, \quad (68)$$

where ε^ν is a unit vector in four dimensions [1]. Equation (68) links the tensor and vector duals of $G^{\nu\rho}$, the antisymmetric field tensor. From Eq. (68),

$$\tilde{G}_\mu\varepsilon^\mu = 0 \quad (69)$$

and this is the field interpretation of [12]

$$\tilde{W}_\mu P^\mu = 0, \quad (70)$$

for the photon, or any other fundamental particle. The latter is defined through the mass and spin invariants $P_\mu P^\mu$ and $\tilde{W}_\mu \tilde{W}^\mu$ respectively. Therefore if \tilde{W}_μ were zero, the particle spin would be zero, in conflict with empirical data. Similarly, the mass and spin invariants of

the classical electromagnetic field are proportional respectively to $\varepsilon_\mu\varepsilon^\mu$ and $\tilde{G}_\mu\tilde{G}^\mu$. These are both zero if the field is massless, but this does not mean that ε_μ and \tilde{G}_μ are zero. These points of fundamental relativity and topology are illustrated in the following development.

Firstly consider a unit lightlike ε^μ proportional to the potential fourvector A^μ considered to be a polar vector proportional through the minimal prescription to the energy momentum four-vector. There is freedom to choose ε^μ as long as $\varepsilon_\mu\varepsilon^\mu = 0$, a condition necessitated by the fact that the electromagnetic field is considered to be concomitant with a massless photon for the sake of argument. This freedom to choose ε^μ is linked to the well known gauge freedom in A^μ , *i.e.*, we are free to choose A^μ to satisfy $A_\mu A^\mu = 0$. There are also conditions to link \mathbf{A} to \mathbf{B} . The first of these is the well known $\mathbf{B} = \nabla \times \mathbf{A}$, which is satisfied by Eq. (64). However, there are other equations that link \mathbf{B} to \mathbf{A} . For example, a particular solution of Eq. (60b) is

$$\frac{\partial}{\partial t}(\mathbf{A} \cdot \mathbf{B}) = 0, \quad \nabla \cdot (A_0\mathbf{B}) = 0, \quad \nabla \cdot (\mathbf{E} \times \mathbf{A}) = 0 \quad (70)$$

one which looks quite different from Eq. (63), but which has the same source, Eq. (60a). Equation (70) is satisfied by Eq. (64), and also by conjugate products of the components therein. For example

$$\nabla \cdot (\mathbf{E} \times \mathbf{A}^*) = 0 \quad (71)$$

Using the usual electromagnetic vacuum relations [1 5]

$$E_0 = cB_0, \quad A_0 = \frac{B_0}{\kappa}, \quad \kappa = \frac{\omega}{c} \quad (72)$$

where κ is the wavevector, ω the angular frequency and c the speed of light, we obtain from the transverse solutions in Eq. (64),

$$\mathbf{E} \times \mathbf{A}^* = -i\frac{c}{\kappa}\mathbf{B} \times \mathbf{B}^* = -\frac{c}{\kappa^3}\mathbf{A} \times \mathbf{A}^*. \quad (73)$$

Equation (72) defines the $\mathbf{B}^{(3)}$ field [1 5],

$$\mathbf{B}^{(3)*} = -\frac{i}{B_0}\mathbf{B} \times \mathbf{B}^* = -i\frac{\kappa}{A_0}\mathbf{A} \times \mathbf{A}^*. \quad (74)$$

Therefore we have established the required link between the conservation equation (60a) and the B cyclics [1 5], which in complex circular notation are written as

$$\mathbf{B}^{(1)} \times \mathbf{B}^{(2)} = iB^{(0)}\mathbf{B}^{(3)*}, \quad (75)$$

in cyclic permutation. From Eq. (73) and (71),

$$\nabla \cdot \mathbf{B}^{(3)} = 0, \quad (76)$$

as required of a magnetic field if it is assumed that there are no magnetic monopoles. Equation (75) is also consistent with the fact that $\mathbf{B}^{(3)}$ is longitudinal and phase free.

Equation (73) establishes the critically important difference between $U(1)$ electrodynamics in which $\mathbf{B}^{(3)} = \mathbf{?0}$ and Poincaré group electrodynamics. In $U(1)$ (Abelian)

electrodynamics, the magnetic field is always the curl of the vector potential; in Poincaré group electrodynamics it can also be the non-Abelian cross product $(-ie/\hbar)\mathbf{A} \times \mathbf{A}^*$ [1 5]. The latter is a conjugate product and is an empirical observable in magneto-optics. It is therefore gauge invariant, *i.e.*, is non-zero in any gauge. To obtain it theoretically in a self consistent way, $U(1)$ gauge theory is replaced by non-Abelian gauge theory. This inference has many consequences throughout field theory, too numerous to develop here. For instance, the quantum mechanical equivalent of the classical $\mathbf{A} \times \mathbf{A}^*$ occurs in non-Abelian quantum electrodynamics (*q.e.d.*) in radiative correction terms. The latter may now be interpreted as establishing the existence of the $\hat{B}^{(3)}$ operator (the photomagnetron [1 5]) to a high degree of precision. The existence of $\hat{B}^{(3)}$ leads also to the acceptance of non Abelian *q.e.d.*, which is still a heuristic theory [12], and must be put on a rigorous basis. More generally $\mathbf{B}^{(3)}$, leads to the development of a non-linear *q.e.d.* in which the artificial removal of infinities (renormalization at all orders) may be rendered obsolete. Photon-photon interaction terms in *q.e.d.* can now be interpreted as interaction between $\hat{B}^{(3)}$ operators on different photons, and this is consistent with the empirical observation of Tam and Happer [5] of interaction between circularly polarized electromagnetic beams. The basic paradox of vanishing classical helicity in Abelian electrodynamics is removed in non-Abelian electrodynamics because the \tilde{G}^μ vector becomes

$$\tilde{G}^\mu = (B_3, 0, 0, B_3), \quad (77)$$

and can be denoted conveniently by the simple, fundamental

$$\tilde{G}^\mu = (B^{(0)}, \mathbf{B}^{(3)}), \quad (78)$$

in the circular basis [1 5]. This result establishes $\mathbf{B}^{(3)}$ as the fundamental spin of the electromagnetic field on the classical level.

The issue is no longer the existence of $\mathbf{B}^{(3)}$, but its future role as the archetypal non-Abelian field in electrodynamics. Field theory has evolved into an intricate unified structure, and into a no less intricate quantum electrodynamics, without ever realizing the existence of the fundamental four-vector $(B^{(0)}, \mathbf{B}^{(3)})$ of the classical electromagnetic sector. The task now is to make amends for this oversight and to find new predictions as a result. The fourvector $(B^{(0)}, \mathbf{B}^{(3)})$ is a latecomer on the classical scene, and the correspondence principle demands that quantum theories produce this new classical result selfconsistently. This process may well result in several new fundamental discoveries, for example linking $\mathbf{B}^{(3)}$ to the existence of

the massive photon proposed by de Broglie and the massive magnetic monopole proposed by Dirac: establishing logically the existence of both.

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www.europa.com/~rsc/physics/B3/evans

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