A Non-Riemannian Universe

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A brief historical introduction to unified field theory is given. Einstein's principle of general covariance which led to General Relativity is contrasted with Weyl's principle of projective invariance leading to a geometrical basis for electromagnetism. Projective invariance can bring quantum theory within the scope of geometrical theories as paths with different quantum numbers can be shown to be projectively related.

An expression for a non-Riemannian affine connection is given and field laws are derived using the connection. It is shown that path integral solutions, when inserted into these field equations, yield characteristic equations with definite eigenvalues. The eigenvalues for the field equations are then related to the eigenvalues of the electromagnetic and gravitational fields. The ratio of the latter can be given the physical interpretation of $(e/m)^2$ for an elementary particle.

The equations of the paths in a non-Riemannian space are written down. It is shown that there is a parallel solution which applies at all points along the path. This parallel solution makes explicit the form of the equations which does not conserve the vectors along the path. A simple transformation applied to the vectors does produce conserved vectors along the path, and enables a solution for the functional form of their evolution. This allows a redshift in the wavelength of light received from great distances without implying a recession of the emitting source. The redshift would be dependent on the path of evolution and its environment.

A conclusion is given that non-Riemannian geometry is capable of generating results of significance and interest at both the elementary particle and cosmological levels. Satisfactory answers to physical problems at both levels would be more likely if there were a coherent philosophical basis to the different branches of theoretical physics.

1. Introduction

Although gravitational phenomena, on the scale of the solar system up to the cosmological level, is held to be encompassed by the theory of General Relativity, there are domains of physical phenomena, like electromagnetism and quantum theory, which cannot be so encompassed. Physicists and astronomers have made a demarcation by saying that quantum theory covers the micro-physical domain, general relativity covers the macro-physical domain and electromagnetism can exist in both. Quantum theorists have attempted to construct a theory of quantum electrodynamics and quantum gravity, but these have not been outstandingly successful because of divergence problems.

Weyl put forward a geometrical basis for the origin of the electromagnetic field in analogy with the way in which Einstein had constructed the General Theory from geometrical considerations (Weyl 1918). General covariance means that physical laws must be independent of the co-ordinate system within which they are formulated. The implementation of this principle leads directly to the theory of gravitation embodied by General Relativity (Einstein 1916). Weyl's principle, that of projective invariance, means that physical laws must be invariant against any change in the scale or gauge system used to define lengths. In Riemannian geometry a length unit fixed at one place is determined everywhere. The equations of the paths in Riemannian geometry, and any other equation involving covariant derivatives, involve the Christoffel three-index symbols, which are not of tensorial nature. General Relativity uses a Riemannian connection consisting of these Christoffel three-index symbols. Weyl supplements this with a non-Riemannian connection which is of a tensorial nature (Eisenhart 1927). This dichotomy has never been satisfactorily resolved in unified field theories: gravitational forces arise from the non-tensorial part of the connection, whereas electromagnetic forces arise from the tensorial part.

Eddington took up Weyl's proposal and generalized the connection still further, though it remained symmetric in the lower indices (Eddington 1923). The Weyl-Eddington theory was successful in providing a purely geometric origin for the electromagnetic field but did not produce an interaction between the electromagnetic field and the gravitational field. No such interaction has yet been observed experimentally but a unified field theory must have such an interaction, or the two field theories are not formally unified. Einstein became convinced that a generalization of the Riemannian connection was necessary, and tried various forms which made the connection non-symmetric in the lower indices (Einstein, 1956). Schrödinger adopted a point of view which was similar to one of Einstein's earlier proposals and developed a theory along similar lines (Schrödinger 1950). In the Einstein-Schrödinger theory the anti-symmetric part of the connection does not affect the geodesics, and to that extent is arbitrary (Hlavaty 1957).

The source of both the gravitational and electromagnetic fields (mass and charge) exists at the quantum level of elementary particles, and these theories have not assimilated quantum theory. It had been shown (Veblen 1922) that a non-Riemannian addition to the Riemannian connection could yield the same paths by re-defining the length parameter along the path. This projective geometry can be used to give a geometrical interpretation to quantum theory, and thus bring it within the scope of unified field theory. The path integral formulation of quantum theory (Feynman and Hibbs 1965) uses an exponential integral of the action to show the evolution of a field along a path. The integral is multiplied by a quantum number, and changing this quantum number will change the scale along the path. But it can be shown that a projective change to the affine connection will change the quantum number, and that the paths are therefore projectively related (Prasad 1975).

Unified field theory seeks to construct elementary particles from the basic geometry constituting the theory. It is quite easy to choose a co-ordinate system in which the Christoffel three-index symbols are zero (Cartesian). The forces arising from geometry are zero in this system, and so the elementary particle would not exist. Particles cannot be allowed to exist or not, depending on the co-ordinate system chosen, so that this suggests the use of a tensorial form for the affine connection. Many investigations have started with a Lagrangian function which embodies invariance principles and used a variation to derive the affine connection (Schrödinger 1950). However, it is possible to reverse this procedure and postulate a simple affine connection and then examine the resulting field laws. The affine connection used is closely related to those proposed by Eisenhart (Eisenhart 1956, papers II and IV), and generates both the gravitational and electromagnetic forces, thus avoiding the dichotomy referred to above.

2. The Field Laws

We take the non-Riemannian affine connection to be

$$L^i_{jk} = k^i h_{jk} \tag{1}$$

where k^i is a propagation vector and the unified field

$$h_{jk} = g_{jk} + f_{jk} \tag{2}$$

is the sum of the gravitational field tensor g_{jk}, which is symmetric and the electromagnetic field tensor f_{jk}, which is anti-symmetric.

This connection, in addition to producing a mutual interaction of the electromagnetic and gravitational fields, also gives a self-interaction of the gravitational field and a self-interaction of the electromagnetic field (Prasad 1981).

Constructing the covariant derivative of the unified field with respect to the affine connection in equation (1), after taking sums and differences to separate out the symmetric and anti-symmetric parts, gives

$$g_{ij|k} = -k^m g_{mj} f_{ik} - k^m g_{im} f_{jk} \tag{3}$$

and

$$f_{ij|k} = -k^m f_{mj} g_{ik} - k^m f_{im} g_{jk} \tag{4}$$

for the mutual interaction equations and

$$g_{ij|k} = -k^m g_{mj} g_{ik} - k^m g_{im} g_{jk} \tag{5}$$

and

$$f_{ij|k} = -k^m f_{mj} f_{ik} - k^m f_{im} f_{jk} \tag{6}$$

for the self-interaction equations. Taking the two equations obtained by symmetric permutation of the indices in each of these four equations results in a total of twelve interaction equations. For the path integral solutions to these equations we adopt for equations (3)

$$g_{ij} = g_{ij}^{(0)} \exp\left[r \int k^i f_{ij} l^j \,\mathrm{d}z \right] \tag{7}$$

and for the equations (4)

$$f_{ij} = f_{ij}^{(0)} \exp\left[r \int k^i g_{ij} l^j \,\mathrm{d}z \right] \tag{8}$$

For the equations (5) we adopt the path integral solutions

$$g_{ij} = g_{ij}^{(0)} \exp\left[r \int k^i g_{ij} l^j \,\mathrm{d}z \right] \tag{9}$$

and for the equations (6)

$$f_{ij} = f_{ij}^{(0)} \exp\left[r \int k^i f_{ij} l^j \,\mathrm{d}z \right] \tag{10}$$

In these four solutions the tangent vector to the path is given by

$$l^i = \frac{\mathrm{d}x^i}{\mathrm{d}z} \tag{11}$$

and $\mathrm{d}z$ is an element of the path length. Substituting these four solutions into the corresponding field equations (3), (4), (5), (6), and imposing the condition of self-consistency on the resulting equations gives, for the mutual interaction equations (3) and (4), the determinant equation

$$\begin{vmatrix} r & 0 & 0 & 0 & -1 & 1 \\ 0 & r & 0 & 1 & 0 & -1 \\ 0 & 0 & r & -1 & 1 & 0 \\ 0 & 1 & -1 & r & 0 & 0 \\ -1 & 0 & 1 & 0 & r & 0 \\ 1 & -1 & 0 & 0 & 0 & r \end{vmatrix} = 0 \tag{12}$$

and for the self-interaction equations (5) and (6) we obtain the determinant equations

$$\begin{vmatrix} r & 1 & 1 \\ 1 & r & 1 \\ 1 & 1 & r \end{vmatrix} = 0 \tag{13}$$

and

$$\begin{vmatrix} r & -1 & -1 \\ -1 & r & -1 \\ -1 & -1 & r \end{vmatrix} = 0 \tag{14}$$

The eigenvalues coming from equations (12), (13), (14), respectively are

$$r_i = 0,\ 0, +\sqrt{3}, +\sqrt{3}, -\sqrt{3}, -\sqrt{3}, i = 1,2,..,6 \tag{15}$$

$$r_j = 1, 1, -2, \quad j = 7, 8, 9 \tag{16}$$

$$r_k = -1, -1, +2, \quad k = 10, 11, 12 \tag{17}$$

By choosing particular paths for the evolution of the fields (Prasad 1993), we can justify projective changes, equivalent to a superposition of states in quantum theory, which enable the quantum numbers for the field variables to be written in the form

$$r_a = a_a + b_a \sqrt{3}, \quad a = 1,2,3,4 \tag{18}$$

where the coefficients a and b are positive or negative integers or zero . Contracting the field equations (3), (4), (5), (6), above gives either one or other of the two equations

$$\left[r_{(a)} G + n_{(b)} + n_{(g)} \right] = 0, \quad b, g = 1,2,3,4 \tag{19}$$

$$\left[r_{(a)} F + l_{(b)} + l_{(g)} \right] = 0, \quad b,g = 1,2,3,4 \tag{20}$$

These two equations establish a link between the quantum numbers attached to the field variables and the eigenvalues for the electromagnetic field given by

$$f_{im} f^{jm} k^i_{(a)} = l_{(a)} d^j_i k^i_{(a)}, \quad a = 1,2,3,4 \tag{21}$$

and for the gravitational field given by

$$g_{im} g^{jm} k^i_{(a)} = n_{(a)} d^j_i k^i_{(a)}, \quad a = 1,2,3,4 \tag{22}$$

Their ratio

$$\frac{-l_{(a)}}{n_{(a)}} = m^2 \tag{23}$$

can be given the physical interpretation of $(e/m)^2$ for an elementary particle.

3. The Equations of the Paths

The equations of the paths can yield formulae which demonstrate an evolution along the path of the vectors which define the path. Taking the covariant derivative of the tangent vector and the propagation vector with respect to the connection in equation (1) gives

$$l^i_{|j} l^j = k^i l^m h_{mj} l^j = k^i f \tag{24}$$

$$k^i_{|j} l^j = k^i k^m h_{mj} l^j = k^i y \tag{25}$$

These two equations have a parallel solution and a non-parallel solution. For present purposes we require only the former

$$l^i y = k^i f \tag{26}$$

The scalars, which then become

$$f = l^m g_{mj} l^j, \quad y = k^m g_{mj} l^j, \tag{27}$$

cannot be zero, but may be very small. Using this solution in equations (24) and (25) gives

$$l^i_{|j} l^j = l^i y \tag{28}$$

$$k^i_{|j} l^j = k^i y \tag{29}$$

Writing these two equations in terms of intrinsic derivatives gives

$$\frac{d}{ds}\left(l^i \right) = l^i y \tag{30}$$

$$\frac{d}{ds}\left(k^i \right) = k^i y \tag{31}$$

It is seen that neither of these vectors is conserved along the path. However, by introducing the variables

$$\bar{l}^i = l^i \exp\left(-\int y \, ds \right) \tag{32}$$

$$\bar{k}^i = k^i \exp\left(-\int y \, ds \right) \tag{33}$$

the equations (28) and (29) become

$$\bar{l}^i_{|j} l^j = 0 \tag{34}$$

$$\bar{k}^i_{|j} l^j = 0 \tag{35}$$

which written in terms of intrinsic derivatives are

$$\frac{d}{ds}\left(\bar{l}^i \right) = 0 \tag{36}$$

$$\frac{d}{ds}\left(\bar{k}^i \right) = 0 \tag{37}$$

These two vectors are conserved along the path and have the solutions

$$\bar{l}^i = l^i_0 = \text{constant} \tag{38}$$

$$\bar{k}^i = k^i_0 = \text{constant} \tag{39}$$

The evolution of the vectors along the path is therefore given, from equations (32) and (33), by

$$l^i = l^i_0 \exp\left(\int y \, ds \right) \tag{40}$$

$$k^i = k^i_0 \exp\left(\int y \, ds \right) \tag{41}$$

where the parallel solution (26) holds at all points along the path. If the wavelength is taken as being proportional to the inverse of the associated wave vector, then the wavelength along the path is given by

$$l = l_0 \exp\left(-\int y \, ds \right) \tag{42}$$

As the redshifted wavelength increases with distance, this indicates a negative value for the integral along the path. No recession is implied for the emitting source. The amount of redshift depends upon the path and its environment. The integrand, given in equation (27) is a scalar which depends on the value of the gravitational field along the path of evolution. If the scalars given in equation (27) are zero, then there is no evolution and no redshift. If the scalars are not zero, then the redshift can be related to the integral along the path of evolution. The mass of the photon is not related to these scalars, but to the eigenvalue in equation (22).

4. Conclusions

The field interaction equations, outlined in section 2, show that it is possible to generate discrete states with differing mass and identifying quantum numbers from purely geometrical considerations. That this result has been obtained within the framework of unified field theory investigations justifies regarding it as the initial stages of a possible theory of elementary particles. The parallel solution to the equations of the paths, outlined in section 3, shows an evolution along a path of the vectors characterizing that path. This provides a possible interpretation for the redshift in light received from great distances which does not imply a recession of the emitting source. It is of significance that it is the same affine connection, and therefore the same geometry, which has produced these two descriptions. Once the affine connection has been chosen the field equations and the equations of the paths are uniquely determined apart from any projective changes.

These two examples show that non-Riemannian geometry can be used to derive results of relevance in physical theory. Of course this is a long way form proving that we live in a non-Riemannian universe. Nevertheless they show that the demarcation of the universe into domains of application is not necessarily valid. That a single geometrical structure can apply at the level of elementary particles and at cosmological distances is, therefore, suggested but not proved. The macro- and micro-physical worlds are linked, as is shown by the relation between the dimensionless constants that can be constructed. The implication is that the geometry of the large-scale universe is linked directly to the geometry of the small-scale world of elementary particles.

If we identify different quantum numbers attached to the same path integral with a different scale, or gauge system, along the path, then quantum theory can be given an interpretation within projective geometry. This would enable all three branches of theoretical physics to have a geometrical interpretation. The compartmentalization of physical theory leads to inconsistent treatments of physical problems, such as quantum electrodynamics as compared with quantum gravity. At present, the philosophical basis of each branch, gravitational theory, electromagnetism and quantum theory, is incompatible with the others. Theoretical physics cannot be classed as a coherent philosophical system while its branches sustain incompatible philosophical viewpoints. If these three branches could be reformulated within a unified and coherent geometrical framework, then there would be a greater chance of obtaining satisfactory answers to the philosophical and physical questions we may pose.

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