Do Various Indices of Galaxy Clustering Describe the Same Physical Property?

Zbigniew Garncarek
Institute of Mathematics
Pedagogical University
Opole, Poland

Konrad Rudnicki
Jagiellonian University Observatory
Krakow, Poland

From a study of values of indices c^2, g^2 and s^2 in the statistical reduction method and the *k*-index in Zwicky's subsequent division method applied to the same population, it is concluded that indices of the same general type of clustering, but formulated in different ways mathematically, sometimes lead to opposite results.

The pattern of clustering of galaxies is by no means intrinsically easy to describe. Numerous efforts to obtain a mathematical formulation of clustering that is both clear and close to reality, from the early models of Neyman to sophisticated contemporary methods, have failed to live up to expectations. Expressions like, "single cluster" of galaxies, "super-cluster", "void", "filament between clusters", *etc.*, are among the fuzziest in the astronomical vocabulary, and are not subject to any strict physical or mathematical definition.

Another unclear notion is the measure of intensity of clustering. In different descriptive methods, different indices are used. One might expect that indices belonging to different mathematical descriptive methods would possess different sensitivities to clustering effects. The conviction still prevails, however, that all of these different measures of intensity of clustering somehow describe, albeit in different ways, the same physical feature of clustering. The following study was performed in order to determine what the real situation in this field is.

What we have done is directly compare four different indices of clustering on the celestial sphere (two dimensions) from two methods based on division of an investigated area into elementary domains (squares). The first is the statistical reduction method developed by Andrzej Zieba. It relies on the notion of fundamental domains, in our case two adjacent elementary domains (for a general definition see Garncarek *et al.* 1988). The second is the k-index method of Fritz Zwicky, which uses division into single elementary domains. In this paper we refrain from comparing indices from other methods, such as the correlation-function-method, percolation method or three-circles-into-one method (for a formal review of methods, see *e.g.* Rudnicki 1988), because they are not based on a division of space into elementary domains, *i.e.* a division of a plane into squares.

The concentration index c^2 and grouping index g^2 are principal indices of the Statistical Reduction Method [S. Zieba 1988, p. 68, or A. Zieba 1975, formulae (5.2) and (6.1)]. The indices are given by the formulae:

$$\begin{aligned} c^2 &= \frac{P_{20}^2 + P_{02}^2}{R_{20}^2 + R_{02}^2} \\ &= \frac{\mathbf{N}}{n(n-1)} \sum_{i=1}^{\mathbf{N}/2} \left[m_1^{(i)} \left(m_1^{(i)} - 1 \right) + m_2^{(i)} \left(m_2^{(i)} - 1 \right) \right] \end{aligned} \tag{1}$$

$$g^2 = \frac{P_{11}^2}{R_{11}^2} = \frac{2\mathbf{N}}{n(n-1)} \sum_{i=1}^{\mathbf{N}/2} m_1^{(i)} m_2^{(i)} \tag{2}$$

(Garncarek *et al.* 1988).

The so-called structural index of the statistical reduction method

$$s^2 = \frac{c^2}{g^2} \tag{3}$$

has also been taken into consideration (Zieba 1988).

The *k*-index of Zwicky's method is defined as follows:

$$k = \frac{S_0}{S_c} \tag{4}$$

where (Zwicky 1957)

$$S_0 = \left[\frac{1}{\mathbf{N}} \sum_{i=1}^{\mathbf{N}/2} \left(m_i - \frac{n}{\mathbf{N}} \right)^2 \right]$$

In the above formulae P_{20}^2 and P_{02}^2, to simplify somewhat, describe the real frequency of fundamental domains with large concentrations of objects in the first or second elementary domain, respectively, P_{11}^2 the frequency of fundamental domains with a quasi-equal number of objects in both elementary domains, $R_{20}^2, R_{02}^2 \left(R_{20}^2 = R_{02}^2 \right), R_{11}^2$ the same for random distribution of objects; *n* denotes the number of objects in the map, $\mathbf{N}$ the number of elementary domains, m_1 the number of objects within the *i*th elementary domain, $m_1^{(i)}, m_2^{(i)}$ respective numbers of objects within the first and second elementary domains of the ith fundamental domain in cases where the method used makes use of the notion of fundamental domain (for a more rigorous explanation see Garncarek *et al.* 1988 and Garncarek *et al.* 1977).

We decided to compare the behaviour of these indices using not just any theoretically (or numerically) calculated set of points but a sample of real extragalactic objects. For a comparison of methods, rather than an investigation of astronomical reality, a two-dimensional distribution is just as good as a three-dimensional distribution: all the investigated indices remain valid with the same descriptive formulae for any number of dimensions. Moreover, in all three-dimensional surveys available at present, the problem of radial coordinates, *i.e.* the problem of transforming distance indicators into distances, emerges and causes additional complications during processing. Thus, for our purposes, we applied the two-dimensional distribution of galaxies as it is given in the Jagellonian Field Catalogue (Rudnicki *et al.* 1973). We chose objects of visibility class 3 in blue (for a strict definition see the preface to the catalogue) because this material presents a fairly "general clustering field" without any particularly distinct, regular individual cluster or any distinct voids.

The results are given in Table 1. This shows values of concentration index, c^2, grouping index, g^2, structural index, s^2, and Zwicky's *k*-index for seven different maps. The first map is a square divided into 144×144 elementary domains of size 2'5×2'5 each. The other maps have been obtained by dividing the same square area into 72×72, 36×36, 18×18, 8×8, 4×4 and 2×2 elementary domains respectively. For each index *x*, the relative variability range

(5)

is given in the table below, where xmax, Xmm~ Xme~ denote the maximal, minimal and mean value of a given index respectively.

From an inspection of the data in Table 1, the following conclusions can be drawn:

1. Among the simple indices of the statistical reduction method in general, the s^2 index and Zwicky's *k*-index show similar variations.
2. The relative variability range can be regarded as a certain measure of sensitivity in specific indices. It increases when going from c^2 to s^2 to k to g^2 . Of course, this is so only in the

Table 1 - Behaviour of different clustering indices for various divisions of the same area

Number of elementary domains	Size of elementary domains	c^2	g^2	S^2	k
144X144	2'.5X2'.5	1.77	4.04	0.44	1.08
72X72	5'X5'	1.47	1.95	0.75	1.18
36X36	10'X10'	1.23	1.51	0.81	1.33
18X18	20'x20'	1.13	1.33	0.85	1.66
8X8	40'X40'	1.06	1.04	1.02	2.12
4X4	1°20*X1 °20'	1.02	1.04	0.99	2.39
2X2	2°40'X2°40'	1.01	0.98	1.03	3.36
relative variability range		0.61	1.80	0.70	1.22

case of average clustering fields. Here the g^2 index proves to be the most sensitive. Other indices may prove to be more sensitive (and hence more useful) for determining particular patterns of clustering.

3. The c^2 and g^2 indices may, as is the case here, vary monotonously in the reverse direction from k and s^2, which shows that the various indices not only possess different sensitivities, but that they differ materially from one another and describe different aspects of physical reality.

All indices considered here are based on the same general idea. They are measures of the deviation from a random distribution of points, as it follows from their definitions. In spite of this, some of these indices increase, some decrease for subsequent divisions of the same area populated by real galaxies. This shows clearly that they measure different features in the distribution of objects.

Conclusion

To say that clustering in a given sample of objects is stronger or weaker has no definite meaning unless a mathematical description and physical interpretation of a specific characteristic of the clustering is given.

References

Garncarek, Z., Kuldewski, J. 1977. *Zeszyty Naukowe WSP*

Opole, Matematyka 20, 127.

Garncarek, Z., Kuklewski, J., Rudnicki, K. 1988. *Acta Cosmologica* 15, 63.

Rudnicki, K. 1988. in *Large Scale Structures of the Universe,* ed. W.C. Seitter, H.W. Durbeck, M. Tacke, p. 306.

Rudnicki, K., Dworak, T.Z., Fin, P., Baranowski, B., Sendrakowski, A. 1973. *Acta Cosmologica* 1, 7

Zieba, A. 1975. *Acta Cosmologica* 3, 75

Zieba, S. 1988. *Acta Cosmologica* 15,85

Zwicky, F. 1957. *Morphological Astronomy.* Berlin: Springer Verlag.